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ABSTRACT

This third unit in the SMSG junior high mathematics series is the teacher's commentary for Unit 1. A time allotment for each of the chapters in Unit 1 is suggested. Then, for each of the chapters in Unit 1, the objectives for that chapter are specified, the mathematics is discussed, some teaching suggestions are provided, the answers to exercises are listed, and sample test questions for that chapter are suggested. (DT)

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School Mathematics Study Group

Mathematics for Junior High School, Volume 1

Unit 3

Mathematics for Junior High School, Volume I

Teacher's Commentary, Part I

REVISED EDITION

Prepared under the supervision of the
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PREFACE

Key ideas of junior high school mathematics emphasized in this text are: structure of arithmetic from an algebraic viewpoint; the real number system as a progressing development; metric and non-metric relations in geometry. Throughout the materials these ideas are associated with their applications. Important at this level are experience with and appreciation of abstract concepts, the role of definition, development of precise vocabulary and thought, experimentation, and proof. Substantial progress can be made on these concepts in the junior high school.

Fourteen experimental units for use in the seventh and eighth grades were written in the summer of 1958 and tried out by approximately 100 teachers in 12 centers in various parts of the country in the school year 1958-59. On the basis of teacher evaluations these units were revised during the summer of 1959 and, with a number of new units, were made a part of sample textbooks for grade 7 and a book of experimental units for grade 8. In the school year 1959-60, these seventh and eighth grade books were used by about 175 teachers in many parts of the country,and then further revised in the summer of 1960.

Mathematics is fascinating to many persons because of its opportunities for creation and discovery as well as for its utility. It is continuously and rapidly growing under the prodding of both intellectual curiosity and practical applications. Even junior high school students may formulate mathematical questions and conjectures which they can test and perhaps settle; they can develop systematic attacks on mathematical problems whether or not the problems have routine or immediately determinable solutions. Recognition of these important factors has played a considerable part in selection of content and method in this text.

We firmly believe mathematics can and should be studied with success and enjoyment. It is our hope that this text may greatly assist all teachers who use it to achieve this highly desirable goal.

The preliminary edition of this volume was prepared at a writing session held at the University of Michigan during the summer of 1959, based, in part, on material prepared at the first SMSG writing session, held at Yale University in the summer of 1958. This revision was prepared at Stanford University in the summer of 1960, taking into account the classroom experience with the preliminary edition during the academic year 1959-60.

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NOTE TO TEACHERS

Based on the teaching experience of nearly 200 junior high school teachers in all parts of the country and the estimates of the authors of the revision (including junior high school teachers), it is recommended that teaching time for Part 1, be as follows:

Chapter	Approximate number of days
1	7
2	15
3	14
4	15
5	12
6	17
7	13
8	13
Total	106

Teachers are urged to try not to exceed these approximate time allotments so that pupils will not miss the chapters at the end of the course. Some classes will be able to finish certain chapters in less than the estimated time.

Throughout the text, problems, topics and sections which were designed for the better students are indicated by an asterisk (*). Items starred in this manner should be used or omitted as a means of adjusting the approximate time schedule.

Chapter 1

WHAT IS MATHEMATICS?

General Remarks

This chapter is intended to give the pupil an appreciation for the importance of mathematics. Its objectives are:

I. To develop an understanding of what mathematics is as opposed to simple computation.

II. To develop an appreciation of the role of mathematics in our culture.

III. To motivate pupils by pointing out the need for mathematicians and for mathematically trained people.

Since this chapter is much different from ordinary textbook material it will need a different treatment. The purpose of the chapter is not to teach many facts or skills, but rather to build an enthusiasm for the study of mathematics. Good attitudes will be built if you use imagination and enthusiasm in getting these objectives across to the pupils. Since the material is not to be taught for mastery, we strongly recommend that no test be given covering the contents of this chapter.

Experience shows that this chapter can be covered within six to eight lessons. Certainly no more than eight days should be devoted to it.

Where seventh graders are in a new school situation and have so many interruptions during the first few days, some teachers may wish to precede this chapter with review exercises which are more familiar to the pupils.

Note that Exercises 1-6, (Page 13) and Exercises 1-7, (Page 14) are suggestions for background study to be carried on throughout the year. These should be begun during the first week, with periodic reports on progress by pupils. Where guidance personnel are available, their services should be solicited to help the class outline a plan of action for the year.

It might be worthwhile to have the pupils read this chapter again at the end of the year. The problems might also be solved again. They should be much easier to solve after the course has been completed.

Encourage the more able students to solve the brainbusters but be ready to help them if they have difficulties. Most pupils will want to puzzle over the brainbusters for a few days. For this reason, only individual help is suggested until the time seems appropriate for general class discussion.

1-1. Mathematics as a Method of Reasoning.

Page 1. It might provide additional challenge to emphasize to the pupils that Exercises 1-1 and Exercises 1-2 are not easy. Moreover, no simple formula for solution can be given. Some of the pupils (and many parents!) will certainly find the problems difficult and time-consuming at this stage. You may not wish to assign all the problems in these two sections.

Answers to Exercises 1-1--Page 2:

1. Two sons cross; one returns. Father crosses; other son returns. Two sons cross.
2. No. They need a boat carrying 225 pounds. Solution as in 1 above.
3. Man takes goose and returns alone. He takes fox and returns with goose. He takes corn across river and returns alone to pick up goose.
4. Yes. This depends on the fact that $8x + 5y = 2$ has solutions in integers, such as $x = -1$, $y = 2$ and $x = 4$, $y = -6$. The first means that if you fill the 5-gallon jug twice and empty it once into the 8-gallon jug, you will have 2 gallons left. The second solution means that if you fill the 8-gallon jug four times and use it to fill the 5-gallon jug 6 times, you will have 2 gallons left. Point out that the first solution is best.

[pages 1-2]

5. Most pupils will need paper and pencil for this one. As they attempt the solution in front of the class it is a good idea to minimize the help from others in the class. If an error is made, another student should be selected to present his solution. It should be pointed out that a crossing necessitates a landing.

   M_1 M_2 M_3 C_1 C_2 C_3

   M_1 C_1 cross river; M_1 returns.
   C_2 C_3 cross river; C_1 returns.
   M_1 M_2 cross river; M_1 C_2 return.
   M_1 M_3 cross river; C_3 returns.
   C_1 C_2 cross river; C_1 returns.
   C_1 C_3 cross river.

6. Balance the two groups of 3 marbles each. If they balance then it is only necessary to balance the remaining two marbles to find the heavy one. If the two groups of 3 marbles do not balance, take the heavier group. Of the 3 marbles in the heavier group balance any 2 marbles. If they balance, the remaining marble is the heaviest one. If the 2 marbles do not balance, the heaviest will be 1 of the 2 on the balance.

7. In solving the problem is it practical to try out all the possible ways the dominoes may be placed on the board? This would be difficult because there are more than 65,536 ways to cover the whole board. The solution may be found in another way:

   How many squares are there altogether on the board? (64)

   How many squares must be covered? (62)

   What is special about the two squares next to each other? (They are of different colors.)

   What is special about the two opposite corners? (They are the same color.)

If you place any number of dominoes on the board can you say anything about the kinds of squares which will be covered? How does this compare with the kinds of squares which you are supposed to cover? Do you have to make even one experiment in order to get the answer to the original problem? Can two squares of the same color be covered with one domino? The answer to this question should help some of the students reason why the solution is impossible.

1-2. Deductive Reasoning.

Page 3. After the concept of deductive reasoning has been introduced, it is still necessary to give the students an understanding of its importance. Examples of scientific advances in which deductive reasoning was used might be Einstein's discovery of the atomic energy formula $E = MC^2$, space travel, satellites, digital computer development, nuclear energy, etc.

Answers to Exercises 1-2 - Page 4:

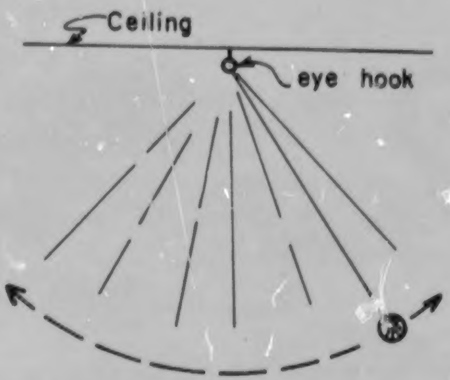
1. After each person receives one pencil, the 5th pencil must go to one of the 4 people.
2. (a) Yes.
   (b) Someone gets at least 3 pencils.
3. 367. (The answer must take leap year into account.)
4. 1. Tell the class that the first two to enroll might be twins.
5. 6.
6. 1.
7. Only 1.
8. 6 committees.
9. 12 committees.

[pages 3-4]

1-3. From Arithmetic to Mathematics.

Page 5. Before considering Gauss's discovery of summing an arithmetic series, you may wish to offer the following as an example of a scientific experiment in which mathematics is used.

Make a pendulum by fastening a weight to a string 40 inches long. Attach the string to an object that will not move. The pendulum should swing freely. Set the pendulum into motion. Count the number of times it swings back and forth in 30 seconds. Shorten the string by 5 inches and repeat this experiment. Shorten again by 5 inches and do it again. Make a table of your observations.



Length of string in inches	No. of swings in 30 seconds
40	?
35	?
30	?

Does there seem to be a relationship between the length of the pendulum and the time? Can you predict the number of swings in 30 seconds if the string is 20 inches long? How does the number of swings depend upon the length of the string?

You have actually repeated an experiment done by Galileo about 400 years ago. Galileo was a famous Italian scientist who lived in the years 1564-1642. He got the idea for the experiment by watching a hanging light fixture swing back and forth. He timed the swing by means of his pulse beat. He was one of the first scientists to show how important it was to investigate problems by the experimental method.

In mathematics we often use the inductive method to discover something. Then we use deductive reasoning to prove that it is true.

John Friedrich Karl Gauss was born in Brunswick, Germany, in 1777. He died in 1855 at the age of 78. The pupils may be interested in noting that his lifetime almost spanned the years from the American Revolution to the Civil War.

Many mathematicians consider Gauss as one of the three greatest mathematicians of all times, the other two being Newton and Archimedes.

In this age of space exploration it is interesting to note that Gauss developed powerful methods of calculating orbits of comets and planets. His interests extended also to such fields as magnetism, gravitation, and mapping. In 1833 Gauss invented the electric telegraph, which he and his fellow worker, Wilhelm Weber, used as a matter of course in sending messages.

In 1807 Gauss was appointed Director of the Göttingen Observatory and Lecturer of Mathematics at Göttingen University. In later years the greatest honor that a German mathematician could have was to be appointed to the professorship which Gauss had once held.

This section deals with Gauss's discovery of the known method of summing an arithmetic series. It dramatizes how some pupils (and mathematicians) apply insight to finding a solution to a problem. Your better students should be told that there are methods other than Gauss's for finding the sum of a series of numbers. Some students might be encouraged to discover methods of their own for adding number series quickly.

The "middle number" method is one that may be used. This scheme can be used for an even or an odd number of integers. The following examples may be used to explain this method to the students who have tried to discover other methods.

Example A. $1 + 2 + 3 + 4 + 5 + 6 + 7 = \ ?$

In this series the middle number (4) is the average of the individual numbers of the series. The sum is the product of the middle number (4) and the number of integers in the series or $4 \times 7 = 28$.

Some pupils may prefer to think of the series as
$(1 + 7) + (2 + 6) + (3 + 5) + 4 = (4 + 4) + (4 + 4) + (4 + 4) + 4 = 7 \times 4 = 28$.

Example B. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \ ?$

In this case the "middle number" is halfway between 4 and 5, or $4\frac{1}{2}$. Then the product $(4\frac{1}{2}) \times 8 = 36$ is seen to give the correct sum.

It may seem more plausible here to write the sum as
$(1 + 8) + (2 + 7) + (3 + 6) + (4 + 5) =$
$(4\frac{1}{2} + 4\frac{1}{2}) + (4\frac{1}{2} + 4\frac{1}{2}) + (4\frac{1}{2} + 4\frac{1}{2}) + (4\frac{1}{2} + 4\frac{1}{2}) = 8 \times 4\frac{1}{2}$.
Clearly, Gauss's method is to be preferred in this case.

Answers to Exercises 1-3--page 6:

1. Another method is this: $2 + 4 = 3 + 3$, $1 + 5 = 3 + 3$.
   That is, the sum is the same as:
   $3 + 3 + 3 + 3 + 3 = 5 \times 3 = 15$.
   This can be called the "averaging method."
2. Either method works. Gauss method: $\frac{8 \times 5}{2}$;
   Averaging method: $5 \times 4 = 20$.
3. $\frac{16 \times 8}{2} = 64$. Here there is an even number of quantities so that the "averaging method" must be modified to give 8 eight's or $8 \times 8 = 64$.
4. (a) 4.
   (b) 9.
   (c) 16.
   (d) The sum of the first "n" consecutive odd numbers equals the square of "n".
   (e) 64.

5. $\frac{24 \times 6}{2} = 72; \ 6 \times 12 = 72.$

6. $\frac{62 \times 10}{2} = 310; \ 31 \times 10 = 310.$

7. $\frac{50 \times 51}{2} = 1275; \ 51 \times 25 = 1275.$

8. Yes, provided that the numbers are in arithmetic progression; that is, there is the same difference between each pair of adjacent numbers.

9. Yes. If we start with 1 there are 200 integers in the series giving us $\frac{(1 + 200)200}{2}$. If we start with 0 there are 201 integers in the series giving us $\frac{(0 + 200)201}{2}$.

   The products of like factors are equal. The method also may be used in a series if we select a number other than 1 or 0 as a starting point. Some of the better students may investigate whether the method works in other number series.

10. (a) If you add 1 to the quantity, the sum up to any number is equal to the next number. Hence, the sum plus 1 is equal to $2 \cdot 256 = 512$. Therefore, the sum is 511.

    (b) This is more in the spirit of Gauss:

    $\text{Sum} = 1 + 2 + 4 + \dots + 256$

    $2 \times \text{sum} = 2 + 4 + \dots + 256 + 512$

    Subtracting: Sum = 511.

11. $\text{Sum} = 2 + (6 + 18 + \dots + 486)$

    $3 \times \text{sum} = 6 + 18 + \dots + 486 + 1458$

    Subtracting: $2 \times (\text{the sum}) = 1458 - 2 = 1456$

    Sum, 728.

1-4. Kinds of Mathematics.

Page 7. Discussion of this section should emphasize the dynamic character of mathematics. It is not a "dead" subject as many parents believe.

It is important also to point out here (and throughout the course) that certain important ingredients are common to all the many varieties of mathematics. The method of logical reasoning, the use and manipulation of abstract symbols, the insistence on precision of thought and clarity of expression, the emphasis on general results--these are some characteristics which need to be stressed whenever possible.

Probability--page 8:

This section gives only a very brief introduction to probability. Although students may become interested at this point and attempt more complex problems, it would be better if they waited. A chapter on probability is included in Volume II. Some supplementary units also are available.

Answers to Exercises 1-4--page 9:

2. One out of four or $\frac{1}{4}$.
3. One out of two or $\frac{1}{2}$.
4. One out of 52 or $\frac{1}{52}$.
5. Four out of 52 or $\frac{4}{52} = \frac{1}{13}$.

   The probability may be thought of as a ratio of

   $$\frac{\text{the number of possible favorable selections}}{\text{the total number of all possible selections}}.$$

   The pupils should be reminded that to say his chances are 1 out of 13 does not mean that he will necessarily draw an ace in the first 13 draws.
6. One out of six. A die has 6 sides and only one side has two dots.
7. There are four possibilities in all, only one of which is favorable. Hence the probability is $\frac{1}{4}$.
8. One out of 36. The more advanced students should reason that there are 36 possible combinations. A table may be constructed to show the possibilities. A possible diagram is the following.

Number of Possibilities	1	2	3	4	5	6	7	8	9 ... 34	35	36
1st die	1	1	1	1	1	1	2	2	2 ... 6	6	6
2nd die	1	2	3	4	5	6	1	2	3 ... 4	5	6

Since there is only one possible way of making two ones the probability is $\frac{1}{36}$.

9. The possibilities are easily enumerated.

Number of Possibilities	1	2	3	4	5	6	7	8
1st coin	H	H	H	H	T	T	T	T
2nd coin	H	H	T	T	H	H	T	T
3rd coin	H	T	H	T	H	T	H	T

For 3 heads to come up, the probability is $\frac{1}{8}$. For exactly 2 heads the probability is 3 out of 8 or $\frac{3}{8}$. For at least two heads the chance is 4 out of 8 or $\frac{1}{2}$. Note that this equals the probability of exactly two heads $(\frac{3}{8})$ plus the probability of 3 heads $(\frac{1}{8})$.

Class Activities 1-6 and 1-7--pages 13-14:

The exercises suggested in Section 1-6 and Section 1-7 should be undertaken as year-long projects to be reported on periodically. Much of this information would be good bulletin-board material. A general aim is to make the pupils alert to the current news relating to mathematics and mathematicians. As the year progresses they should gain an increasing appreciation of the important role mathematical thought is playing in our present civilization.

The National Science Foundation, Washington 25, D.C., publishes pamphlets which contain information on the number of mathematicians and where they are employed.

A survey of college requirements in the vocations students may choose should be especially valuable at this stage. We hope it may ease the difficult task of effective guidance in choosing their high school courses.

One word of caution is perhaps warranted here. The purpose of these sections on mathematics today is to call attention to its central role in the pupil's daily life. The necessity for a minimum knowledge of mathematics is to be stressed. It is not the intention to recruit people for careers as mathematicians.

1-8. Mathematics for Recreation.

Page 15. Bring out the fascination of mathematics as a leisure activity or hobby. Encourage students in finding recreational mathematics from the books available at school and from current magazines or rotogravure sections of newspapers.

Choose such problems now and then, throughout the year, at a time when the class needs a change of pace. These kinds of problems can be used profitably with the class period before a lengthy vacation.

Discussion of Königsberg Bridges Problem.

By experimenting you can show that it is impossible to pass just once over every bridge if there are more than two points where an odd number of routes come together. Since there are four points where an odd number of routes come together, this makes it impossible to walk over each bridge once and only once. You may wish to consult the supplementary unit on the Königsberg Bridges for further ideas.

Answer to Exercise 1-8--page 16:

1. The first figure can be drawn if you start at either of the vertices where an odd number of segments come together. The second figure has no such vertices so it can be drawn by starting at any vertex. The third figure has four vertices where an odd number of segments come together so it cannot be drawn without lifting your pencil or retracing a line.

1-9. Highlights of First-Year Junior High School Mathematics. Page 16. This section is intended to give the student a general overview of the scope of the first-year course in junior high school mathematics. It is intended to broaden his ideas of mathematics by discussing very briefly the topics that will be studied. This, together with the introduction of logical reasoning and probability earlier in this chapter, will help dispel the general idea that mathematics is computation only.

Reading this chapter again later in the year may strengthen the student's understanding of what mathematics is.

Chapter 2

NUMERATION

Introduction

For this unit little background is needed except familiarity with the number symbols and the basic operations with numbers. The purpose of the unit is to deepen the pupil's understanding of the decimal notation for whole numbers, especially with regard to place value, and thus to help him delve a little deeper into the reasons for the procedures, which he already knows, for carrying out the addition and multiplication operations. One of the best ways to accomplish this is to consider systems of number notations using bases other than ten. Since, in using a new base, the pupil must necessarily look at the reasons for "carrying" and the other mechanical procedures in a new light, he should gain deeper insight into the decimal system. A certain amount of computation in other systems is necessary to "fix" these ideas, but such computation should not be regarded as an end in itself. Some of the pupils, however, may enjoy developing a certain proficiency in using new bases in computing.

Perhaps the most important reason for introducing ancient symbolisms for numbers is to contrast them with our decimal system, in which not only the symbol, but its position, has significance. It should be shown, as other systems are presented, that position has some significance in them also. The Roman System has a start in this direction in that XL represents a different number from LX, but the start was a very primitive one. The Babylonians also made use of position, but lacked a symbol for zero until about 200 B.C. The Babylonian symbol " ≪ " denoted the absence of a figure but apparently was not used in computation. The numeral zero is necessary in a positional system. In order for the pupils to appreciate the important characteristics of our system of writing numbers, the following table may be discussed.

	Base	Place Value	Zero
Egyptian	Ten	No	No
Babylonian	Sixty	Yes	Limited meaning
Roman	Varied	No, but it has positional value	No
Decimal	Ten	Yes	Yes

Pupils should not be expected to memorize ancient symbolism. It is recommended that little time be spent on the use of the symbols themselves.

It is especially important to distinguish between a number and the symbols by which it is represented. Some of the properties usually connected with a number are really properties of its notation. The facts that, in decimal notation, the numeral for a number divisible by 5 ends in 5 or 0, and that $\frac{1}{3}$ has an unending decimal equivalent (0.333...), are illustrations. Most of the properties with which we deal are properties of the numbers themselves, and are entirely independent of the notation in which they are represented. Examples of such properties of numbers are: $2 + 3 = 3 + 2$; the number eleven is a prime number; and six is greater than five. The distinction between a number and the notation in which it is expressed should be emphasized whenever there is opportunity.

An attempt has been made to use "number" and "numeral" with precise meaning in the text. For example, "numerals" are written, but "numbers" are added. A numeral is a written symbol. A number is a concept. Later in the text it may be cumbersome to the point of annoyance to speak of "adding the numbers represented by the numerals written below." In such case the expression may be elided to "adding the numbers below."

At several points, numbers are represented by collections of x's. Exercises of this kind are important, because they show the role of the base in grouping the x's, as well as the significance of the digits in the numeral for the number.

Suggested Time Schedule

It is important that only enough time be spent on the various sections to secure the understandings desired. The historical symbols themselves are not important. Neither are the numerals in other bases valuable in themselves, but the ideas that they help to clarify are important.

Familiarity with the subject matter is an important factor in a smooth presentation. Teachers report that a second experience with this material is much easier than the first. The lesson moves more rapidly, apparently, as the teacher gains confidence in the subject matter presented.

Homogeneously grouped classes undoubtedly will alter the suggested schedule since the more able students can complete the chapter in less time while less able students may require a considerably longer period of time on various sections. The following schedule may then be adapted to local needs, taking into consideration the length of class periods and other factors. It should be remembered that extra time spent on this chapter will necessarily reduce the number of days available for later important chapters.

Sections	Days
1	1
2	1
3	1-2
4	2
5	3
6	2
7	1
8	1
9	2
Test	1
Total	15-16

Fifteen to sixteen days should be sufficient for the chapter and a test on the chapter.

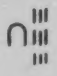
2-1. History of Numerals.

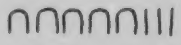
Page 21. The purpose of the historical material is to trace the continuing need for convenient symbols and for a useful way of writing expressions for numbers. The idea of "one-to-one" correspondence is introduced. It is developed later and should not be defined here where the emphasis is upon numerals rather than upon number. Egyptian symbolism is introduced to familiarize the pupils with one of the first important systems of notation. Do not consume an excessive amount of time in discussing the Egyptian or Babylonian systems.

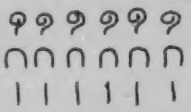
Page 22. The Babylonians were among the first to use place value. The base sixty system is mentioned because of later reference to it, particularly in measurement. There is evidence that the Babylonians also used symbols like ○) but there is no need to introduce these to the pupils. No pupil should be required to memorize ancient symbolism except in the case of Roman numerals.

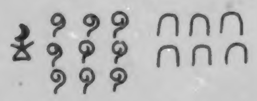
Page 23. The Roman system may be stressed because of its continued use. Note that the subtracting principle was a late development. It may be pointed out that computation in ancient symbolism was complex and sometimes very difficult. Because of this, various devices were used, such as the sand reckoner, counting table, and abacus. After decimal numerals became known, algorithms were devised and people were able to calculate with symbols alone. There was much opposition in Europe to the introduction and use of Hindu-Arabic numerals, especially on the part of the abacists. As the new system became accepted, the abacus and other computing devices slowly disappeared in Europe.

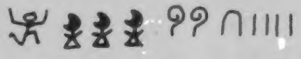
Answers to Exercises 2-1--Page 24:

1. (a) 

(b) 

(c) 

(d) 

(e) 

2. ∩∩|| , ∩|∩| , ||∩∩ ,
∩||∩ , |∩∩| , |∩|∩

3. (a) 200,105 (c) 1029
(b) 2052 (d) 1,100,200

4. (a) ▼▼▼ (b) <<▼▼ (c) <<<<<▼
▼▼▼
▼▼▼

5. (a) 15 (b) 37 (c) 55

6. (a) 29 (e) 666
(b) 61 (f) 500,000
(c) 90 (g) 1492
(d) 105

7. (a) XIX (d) MDCXC
(b) LVII (e) $\overline{M}$
(c) DCCCLXXXVIII (f) $\overline{XV}$

8. (a) 7 (b) 2 (c) 7 (d) 10

9. (a) No. Since X represents a smaller number than C, XC means 100 - 10 or 90 while CX means 100 + 10 or 110.
(b) Yes, since the position indicated whether the numbers are to be added or subtracted.

10. (a) Three.
(b) One hundred eleven.
(c) In the Roman system, the symbol shows the number of units to be added while our system shows the number of groups and each group has a different number of units. The decimal system involves place value while the Roman system does not.

11. (a)
$$\begin{array}{r}1709\\ \underline{654}\\ 2363\end{array}$$
or MMCCCLXIII

(b)
$$\begin{array}{r}2640\\ \underline{1408}\\ 4048\end{array}$$
or MMMMXLVIII

[pages 24-25]

2-2. The Decimal System.

Page 26. The illustration of grouping in tens suggests a method useful in mental calculation as 37 + 62 is 3 tens + 6 tens + 7 + 2 which is 99. Note that parentheses are used to show that certain combinations are to be considered as representing a single numeral. In a later chapter a different use of parentheses will be discussed in greater detail. Emphasize the value represented by a digit and the value of position in decimal notation.

Emphasize the importance of the invention of a useful system which lends itself easily to calculation. The efficiency of the decimal system lies in a combination of factors.

1. Only a few symbols are needed, no matter how large or small the number expressed. Some students may observe that we use ten symbols while the Babylonians used just two and the Egyptians and the Romans each used 7. However, in the decimal system no additional symbols are ever needed as larger numbers are introduced; this is not true of the other systems.
2. Place value in which each position corresponds to a power of the base is of importance in a system used for calculation.
3. The concept of zero as a place holder is essential in the development of a place value system.

Page 28. The reading and writing of numerals may be treated as a review, or if needed, as a thorough study, depending on the needs of pupils. Some pupils may know and understand this material completely. Others may have a very limited proficiency in this area.

A class discussion might deal with the following:

Suppose we used systematic names for numerals such as "two tens" for twenty, "two tens, one" for twenty-one, "ten, one" for eleven, and "ten, two" for twelve. How many different, basic words would be needed to name all counting numbers up to a trillion? (Remember that a numeral like "one hundred, three tens, six" is made up of basic words used many times in other numerals.) There are fifteen essential words: "one, two, three,......, ten, hundred, thousand, ..., trillion."

Answers to Exercises 2-2. Page 29.

1. Ten; 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
2. (1) units, (2) tens, (3) hundreds, (4) thousands, (5) ten thousands, (6) hundred thousands, (7) millions, (8) ten millions, (9) hundred millions.
3. (a) three hundred.
   (b) three thousand five.
   (c) seven thousand one hundred nine.
   (d) fifteen thousand fifteen.
   (e) two hundred thirty-four thousand.
   (f) six hundred eight thousand fourteen.
   (g) one hundred thousand nine.
   (h) four hundred thirty thousand one.
   (i) nine hundred ninety-nine thousand nine hundred ninety-nine.
   (Note: Only the tens numbers are hyphenated, as "twenty-three", etc.)
4. (a) seven million thirty-six thousand two hundred ninety-eight.
   (b) nine trillion three hundred billion seven hundred eight million five hundred thousand.
   (c) twenty billion three hundred million four hundred thousand five hundred.
   (d) nine hundred billion.
5. (a) 159
   (b) 503
   (c) 6857
   (d) 3,070,013
   (e) 4,376,007,000
   (f) 20,010
   (g) 9,015,200
6. 99,999 means $(9 \times 10,000) + (9 \times 1000) + (9 \times 100) + (9 \times 10) + (9 \times 1)$
   ninety-nine thousand nine hundred ninety-nine

7. 100,000 means $(1 \times 100{,}000) + (0 \times 10{,}000) + (0 \times 1000) + (0 \times 100) + (0 \times 10) + (0 \times 1)$. One hundred thousand.

2-3. Expanded Numerals and Exponential Notation.

Page 30. Exponents are introduced here in a situation which shows clearly their usefulness for concise notation. Furthermore, their use serves to emphasize the role of the base and of position. This role will be more fully utilized in the sections to follow.

The Celts and Mayans used twenty as a base probably because they used their toes as well as their fingers in counting. The special name sometimes used for twenty is "score." Some Eskimo tribes count by five using the fingers of one hand.

Answers to Exercises 2-3. Page 31.

1. Ten to the first power, ten to the second power (or ten square), ten to the third power (or ten cube), ten to the fourth power, ten to the fifth power.
2. (a) 3^5
   (b) 2^4
   (c) 6^6
   (d) 25^3
   (e) 5^6
   (f) 4^2
   (g) 279^5
   (h) 16^1
3. (a) three
   (b) seven
   (c) two
   (d) ten
   (e) An exponent of one indicates the value of the base. In a strict sense this 5 is not a factor.
   (f) five
4. (a) $4 \times 4 \times 4$
   (b) $3 \times 3 \times 3 \times 3$
   (c) $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$
   (d) $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$
   (e) $33 \times 33 \times 33 \times 33 \times 33$
   (f) $175 \times 175 \times 175 \times 175 \times 175 \times 175$
5. The exponent tells how many times the base is taken as a factor.

[pages 29-31]

6. (a) $3 \times 3 \times 3 = 27$
   (b) $5 \times 5 = 25$
   (c) $4 \times 4 \times 4 \times 4 = 256$
   (d) $2 \times 2 \times 2 \times 2 \times 2 = 32$
   (e) $6 \times 6 = 36$
   (f) $7 \times 7 \times 7 = 343$
   (g) $8 \times 8 = 64$
   (h) $9 \times 9 = 81$
   (i) $10 \times 10 \times 10 = 1000$
   (j) $3 \times 3 \times 3 \times 3 = 81$
   (k) $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$
   (l) $4 \times 4 \times 4 \times 4 \times 4 = 1024$

7. (a) 4^3 means 64. 3^4 means 81.
   (b) 2^9 means 512. 9^2 means 81.

8. (a) $(4 \times 10^2) + (6 \times 10^1) + (8 \times 1)$
   (b) $(5 \times 10^3) + (3 \times 10^2) + (2 \times 10^1) + (4 \times 1)$
   (c) $(7 \times 10^3) + (0 \times 10^2) + (6 \times 10^1) + (2 \times 1)$
   (d) $(5 \times 10^4) + (9 \times 10^3) + (1 \times 10^2) + (2 \times 10^1) + (6 \times 1)$
   (e) $(1 \times 10^5) + (0 \times 10^4) + (9 \times 10^3) + (1 \times 10^2) +$
   $(8 \times 10^1) + (0 \times 1)$

9.

10^{10}	10,000,000,000	ten billion
10^9	1,000,000,000	one billion
10^8	100,000,000	one hundred million
10^7	10,000,000	ten million
10^6	1,000,000	one million
10^5	100,000	one hundred thousand
10^4	10,000	ten thousand
10^3	1,000	one thousand
10^2	100	one hundred
10^1	10	ten

10. The exponent of the base "10" tells how many zeros are written to the right of the "1" when the numeral is written in the usual way.

11. (a) 10^3 (b) 10^5 (c) 10^6 (d) 10^8

12. 10^{100}. (It may be pointed out to the pupils that 1^{100} is 1.)

13. 100, 10, 1. Some discussion might be devoted to the meaning given to 10^0.. This point need not be stressed at this time, but it will be used in later chapters.

2-4. Numerals in Base Seven.

Page 33. The purpose of teaching systems of numeration in bases other than ten is not to produce facility in calculating with such systems. A study of an unfamiliar system aids in understanding a familiar one, just as the study of a foreign language aids us in understanding our own. The decimal system is so familiar that its structure and the ideas involved in its algorithms are easily overlooked. In this section attention is focused on numerals, rather than on numbers.

Page 34. Questions may arise about the notation for a numeral to base seven. We do not write "15_7" because the symbol "7" does not occur in a system of numeration to this base. Replacing the numeral by the written word emphasizes this fact.

Later in the chapter and in succeeding chapters, some classes may agree to indicate the base, as different systems of numeration are introduced, by a numeral in decimal notation. They may agree that they will regard the subscript in this case as always based on ten. Thus they may write for 25_{seven} the expression 25_7; for 18_{twelve} the expression 8_{12}; and for 110_{two}, 110_2.

After the pupils have had practice in grouping by sevens, introduce counting. Pupils enjoy counting in turn, and helping each other as 30_{seven}, 40_{seven}, and like numerals arise. Have them fill in missing parts of the list on page 35 orally with perhaps a recorder at the board.

Page 35.

Numerals (in base seven) from 21_{seven} to 202_{seven}

	30	40	50	60	100	110	120	130	140	150	160	200
21	31	41	51	61	101	111	121	131	141	151	161	201
22	32	42	52	62	102	112	122	132	142	152	162	202
23	33	43	53	63	103	113	123	133	143	153	163	
24	34	44	54	64	104	114	124	134	144	154	164	
25	35	45	55	65	105	115	125	135	145	155	165	
26	36	46	56	66	106	116	126	136	146	156	166	

At 66_{seven} you may wish to say: "This is the number of states we had in the United States before Alaska became a state. How many states did we have after Alaska and before Hawaii? How shall we write this number in base seven numerals? We have gone as high as we can in the 'one' place and in the 'seven' place. What is one more than 66_{seven}? What do you do when you reach 99 in the decimal system?" When the pupils understand that after 66_{seven} comes 100_{seven}, ask them, "How many states are there when we include Hawaii?" Pupils may read 101_{seven} as "one, zero, one, base seven."

Have the pupils continue to count orally until they reach 202_{seven}.

It is usually helpful to keep the chart on page 35 on the board during the time this section is studied. Some teachers emphasize the meaning of exponents by writing the chart in two ways:

seven^4 (seven × seven × seven × seven)		seven^3 (seven × seven × seven)
seven^2 (seven × seven)	seven^1 (seven)	one (one)

It is suggested that alternate exercises in this list be discussed and answered in class as a group undertaking. The pupils should then be ready to attempt the remaining exercises without further help.

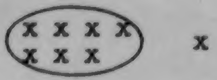
Answers to Exercises 2-4. Page 37.

1. (a) 13_{seven}

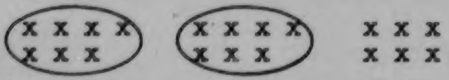
   (b) 24_{seven}

   (c) 116_{seven}

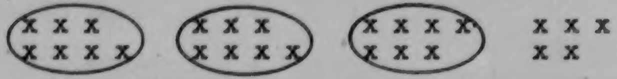
2. (a)



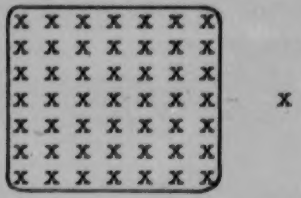
(b)



(c)



(d)



3. (a) (3 × seven) + (3 × one) = 24
   (b) (4 × seven) + (5 × one) = 33
   (c) (1 × seven × seven) = 49
   (d) (5 × seven × seven) + (2 × seven) + (4 × one) = 263

4. (a) 10_{seven} (d) 163_{seven}

   (b) 11_{seven} (e) 1000_{seven}

   (c) 55_{seven} (f) 1010_{seven}

5. (a) 560_{seven} The 6 means 6 sevens

   (b) 56_{seven} The 6 means 6 ones

   (c) 605_{seven} The 6 means 6(seven × seven)'s or 6(forty-nine)'s

   (d) 6050_{seven} The 6 means 6(seven × seven × seven)'s or 6(three hundred forty-three)

6. seven^4 or seven to the fourth power

7. The product of 9 sevens or (7^9).

8. 132_{seven}

9. 452_{seven}.

10. 205_{ten}.

11. Neither. They are equal.

12. (a) Yes. $30_{ten} = (3_{ten} \times ten) + (0 \times 1)$

    (b) No. When 241_{ten} is divided by 10_{ten} there is a non-zero remainder.

    (c) If the units digit is zero the number is divisible by ten; otherwise it is not divisible by ten.

13. $30_{seven} = 21_{ten}$ is not divisible by ten.

    $60_{seven} = 42_{ten}$ is not divisible by ten.

14. (a) It has a remainder of zero when divided by seven.

    (b) Yes. $(3 \times seven) + (0 \times 1)$ is divisible by seven; Remainder is 0.

15. No. $(3 \times 7) + (1 \times 1) = 22$ is not divisible by seven.

16. A number written in base seven is divisible by seven when the units digit is zero.

17. 24_{ten} and 68_{ten} are divisible by two. A number not divisible by two is called an odd number. A number divisible by two is called an even number.

18. 11_{seven} is even.

    No; you cannot tell merely by glancing at the numerals. You could tell by converting each number to base ten. There is another method which is shorter and has much value in teaching divisibility ideas; it may be a bit advanced for grade seven. For example,

$$\begin{aligned} 12_{seven} &= (1 \times 7) + (2 \times 1) = 1 \times (6 + 1) + (2 \times 1) \\ &= (1 \times 6) + (1 \times 1) + (2 \times 1) \\ &= (1 \times 6) + [(1 + 2) \times 1] \end{aligned}$$

The first term is divisible by two but the second term is not divisible by two; hence the sum is not divisible by two. Note that the digit in the units place of the last expression in the display $1 + 2$, is the sum of the digits of 12_{seven} and this sum is not divisible by two. This is a general rule for base seven numerals.

[page 37]

19. They use seven symbols and seem to have a place value system with base seven. They appear to use |, ∠, △, □, ⧅, ⊠, for 1, 2, 3, 4, 5, 6 and — for zero. ∠∠ follows ∠| .

20. See discussion problem 18.

2-5. Computation in Base Seven.

Page 39. Be sure that pupils understand the construction of the addition table for base ten.

Addition, Base Ten

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Pupils should be helped to observe the symmetry of the table with respect to the diagonal. They will notice that 8 + 6 = 6 + 8, for example, and that this is true for any pair of numbers. Later they will learn that this is the commutative property of addition. It is suggested that the word commutative not be used at this time.

If pupils know the facts, no time should be wasted on the table after its characteristics have been discussed.

Page 40.

Addition, Base Seven

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

There is no value in memorizing this table. The process is more important than the facts. The point to be emphasized is that numbers and number properties are independent of the numerals or symbols used to represent the numbers. Commutativity holds in base seven as well as base ten because it is a property of numbers, not numerals.

Answers to Exercises 2-5a.

1. (a) 11 (b) 17

3. (a) Yes
   (b) By reading each result from the table and noting results.
   (c) Chart is symmetric with respect to the diagonal.
   (d) 55 different combinations; just a bit over half the total number of combinations.

5. (a) 28 different combinations. Fewer than 49 because of the commutative law of addition.
   (b) In base seven because there are fewer.
   (c) They are equal since $9 = 12_{\text{seven}}$.

Addition in base seven is undertaken to clarify addition in decimal notation. Some of the newer elementary school textbooks prefer to use the word "change" or "regroup" rather than "borrow" since the first two words seem to describe the actual process better than the last.

Point out to the pupils that in adding in base ten it is often necessary to regroup ten ones as one ten, whereas in base seven we regroup seven ones as one seven.

Page 42. As pupils use the table in subtraction, they may observe that subtraction is the inverse of addition.

Some of the exercises in addition and subtraction are written in horizontal fashion in preparation for future work in algebra.

Answers to Exercises 2-5b--Page 43.

1. (a) 56_{seven}
   $(19 + 22 = 41)$
   (b) 110_{seven}
   $(41 + 15 = 56)$
   (c) 300_{seven}
   $(109 + 38 = 147)$
   (d) 620_{seven}
   $(91 + 217 = 308)$
   (e) 241_{seven}
   $(33 + 94 = 127)$
   (f) 1266_{seven}
   $(199 + 290 = 489)$
   (g) 1553_{seven}
   $(327 + 299 = 626)$
   (h) 14562_{seven}
   $(2189 + 1873 = 4062)$
   (i) 6441_{seven}
   $(2160 + 123 = 2283)$
   (j) 1644_{seven}
   $(327 + 342 = 669)$
   (k) $14{,}654_{seven}$
   $(1917 + 2189 = 4106)$

2. (a) 2_{seven} (b) 4_{seven} (c) 4_{seven}

3. (a) 2
   $(7 - 5 = 2)$
   (b) 36
   $(47 - 20 = 27)$
   (c) 163_{seven}
   $(98 - 4 = 94)$
   (d) 151_{seven}
   $(91 - 6 = 85)$
   (e) 6
   $(32 - 26 = 6)$
   (f) 506_{seven}
   $(323 - 72 = 251)$

(g) 203_{seven}

(247 - 146 = 101)

(h) 406_{seven}

(1715 - 1513 = 202)

(i) 552_{seven}

(319 - 37 = 282)

(j) 36_{seven}

(74 - 47 = 27)

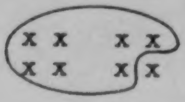
(k) 1254_{seven}

(1261 - 781 = 480)

(l) 54_{seven}

(136 - 97 = 39)

4. (a)



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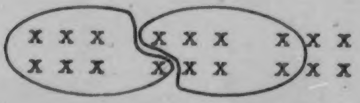

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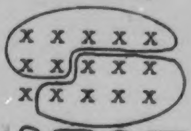

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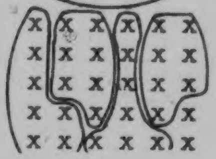


Page 44.

Multiplication, Base Ten

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Answers to Exercises 2-5c. Page 45.

1. Study of this table should emphasize the following:
   (a) The product of 0 and any number is zero.
   (b) The product of 1 and any number is the number itself.

2. The order in multiplication does not affect the product. This is indicated by the fact that the parts of the table on opposite sides of the diagonal line are alike.

3. Multiplication, Base Seven

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

Study of this table is valuable for the additional insight it affords into the understanding of multiplication. There is no value in memorizing it. The table may be used to emphasize that division is the inverse of multiplication.

4. Since multiplication combinations are needed only up to 6 × 6 instead of up to 9 × 9, multiplication is easier to learn in base seven than in base ten.

5. (a) Both parts are alike.
   (b) $3_{seven} \times 4_{seven} = 4_{seven} \times 3_{seven}$; this is an illustration of the fact that multiplication is commutative.

6. 2835; 18,675; 2,017,372; 697,226; 3,981,354.

Page 47. Since division is the most demanding operation, it is suggested that teachers regard the topic as optional and do only as much as they judge appropriate, in class discussion. Pupils may need help in learning how to use the multiplication table to find division facts. Exercises are included for those pupils who wish to attempt them.

Working with base ten and base seven numerals, using the tables, and changing from base seven to base ten, provide many opportunities for needed drill in the four operations. Teachers may wish to observe the kinds of drill needed and to devise additional exercises of the appropriate type.

Answers to Exercises 2-5d.

1. (a) 45_{seven}
   (b) 222_{seven}
   (c) 1116_{seven}
   (d) 3325_{seven}
   (e) 3464_{seven}
   (f) $443{,}115_{seven}$
   (g) $106{,}533_{seven}$
   (h) $5{,}511{,}426_{seven}$
   (i) $125{,}150_{seven}$
   (j) $1{,}660{,}101_{seven}$

2. (a) 5_{seven}
   (b) 62_{seven}
   (c) 421_{seven} with a remainder of 2_{seven}.
   (d) 123_{seven} with a remainder of 12_{seven}.

3. (a) $(4 \times 7 \times 7) + (0 \times 7) + (3 \times 1) = 199_{ten}$
   (b) $(1 \times 10 \times 10) + (8 \times 10) + (9 \times 1) = 189_{ten}$

4. 403_{seven}

5. (a) 66_{seven}
   (b) 123_{seven}
   (c) 1061_{seven}
   (d) 1205_{seven}

6. (a) 4_{seven} (b) 26_{seven} (c) 334_{seven}

7. Room 123; book 7; 15 chapters; 394 pages; 32 pupils; 5 days; 55 minutes; 13 girls; 19 boys; 11 years old; 66 inches or 5 feet 6 inches tall.

2-6. Changing from Base Ten to Base Seven.

Page 48. Pupils in general find it easier to change from base seven to base ten numerals than the reverse. This section gives much detail in order to provide help to pupils who may have trouble.

Ask for the highest power of seven which is contained in the number given in base ten numeration. For example, consider 634_{10}. Is 7^4 or (2401_{10}) contained in 634_{10}? Is 7^3 or (343_{10})? After we have taken as many 343_{10}'s as possible from 634_{10}, how much remains? The next power of 7 is 7^2. How many 49_{10}'s are contained in 291_{10}? Finally, how many 7's, and how many 1's are left?

Page 51. A second method of changing from base ten to base seven numerals is developed in Exercises 3, 4, 5 on Page 51.

$$\begin{array}{rrl}
10 & \underline{|1958} & \\
10 & \underline{|195} & \text{remainder } 8 \\
10 & \underline{|19} & \text{remainder } 5 \\
10 & \underline{|1} & \text{remainder } 9 \\
 & 0 & \text{remainder } 1
\end{array}$$

The number is 1958

$$\begin{array}{rrl}
7 & \underline{|524} & \\
7 & \underline{|74} & \text{remainder } 6 \\
7 & \underline{|10} & \text{remainder } 4 \\
7 & \underline{|1} & \text{remainder } 3 \\
 & 0 & \text{remainder } 1
\end{array}$$

The number is 1346 base seven

Some pupils will see that the 6 ones are found first when 74 groups of 7 are taken away. These groups of 7 are then put together in groups of 7 sevens with 4 groups left for the numeral in the 7 place. The process is repeated to find the digits for successive places in the numeral.

Answers to Exercises 2-6. Page 50.

1. (a) $50_{ten} = (1 \times \text{seven}^2) + (0 \times \text{seven}) + (1 \times \text{one})$

$= 101_{seven}$

(b) $145_{ten} = (2 \times \text{seven}^2) + (6 \times \text{seven}) + (5 \times \text{one})$

$= 265_{seven}$

(c) $1024_{ten} = (2 \times \text{seven}^3) + (6 \times \text{seven}^2) + (6 \times \text{seven}) + (2 \times \text{one}) = 2662_{seven}$

2. (a) 15_{seven}
   (b) 51_{seven}
   (c) 62_{seven}
   (d) 104_{seven}
   (e) 431_{seven}
   (f) 3564_{seven}

3. Q = 195 R = 8
   Q = 19 R = 5
   Q = 1 R = 9
   Q = 0 R = 1

4. Q = 74 R = 6
   Q = 10 R = 4
   Q = 1 R = 3
   Q = 0 R = 1

   $524_{\text{ten}} = 1346_{\text{seven}}$; the digits of the base seven numeral are the remainders which have just been obtained.

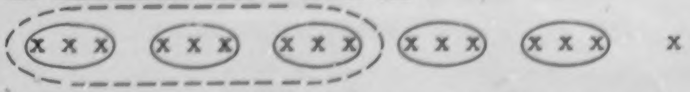
5. Divide by seven and continue to divide each quotient by seven. The digits in the numeral sought will be the remainders in order with the first remainder in the "one" place.

6. (a) 1161_{ten}
   (b) 275_{ten}
   (c) 654_{seven}
   (d) 462_{seven}
   (e) 1116_{seven}
   (f) 3
   (g) 462_{seven}

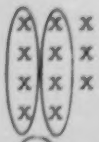
2-7. Numerals in Other Bases.

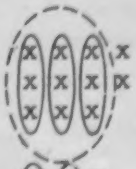
Page 52. Bring out the idea that the base of the system that we use is "ten" for historical rather than mathematical reasons. Some mathematicians have suggested that a prime number such as 7 has certain advantages. The Duodecimal Society of America, 20 Carlton Place, Staten Island 4, New York supports the adoption of twelve as the best number base. Information about the duodecimal system is furnished by this society on request. Exercises in other number bases help establish an understanding of what a positional, power system of numeration is.

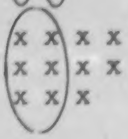
Answers to Exercises 2-7. Page 54.

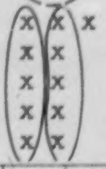
1. 

   (a) 5 groups of three and 1 left over.
   (b) No. Only the digits "0", "1", and "2" are used in the base three system. "5" is not one of these.
   (c) (1 group of three^2) + (2 groups of three) + (1 left over).
   (d) $16_{ten} = 121_{three}$.

2. (a)  , 11_{ten}

   (c)  , 11_{ten}

   (b)  , 11_{ten}

   (d)  , 11_{ten}

3.

Base Ten	0	1	2	3	4	5	6	7	8	9	10
Base Five	0	1	2	3	4	10	11	12	13	14	20

Base Ten	11	12	13	14	15	16	17	18	19	20
Base Five	21	22	23	24	30	31	32	33	34	40

Base Ten	21	22	23	24	25	26	27	28	29	30
Base Five	41	42	43	44	100	101	102	103	104	110

4. (a) two (b) two (c) two (d) two

5. (a) $(2 \times 36) + (4 \times 6) + (5 \times 1) = 101$
   (b) $(4 \times 25) + (1 \times 5) + (2 \times 1) = 107$
   (c) $(1 \times 27) + (0 \times 9) + (0 \times 3) + (2 \times 1) = 29$
   (d) $(1 \times 64) + (0 \times 16) + (2 \times 4) + (1 \times 1) = 73$

Other answers are acceptable, i.e., $(2 \times 6^2) + (4 \times 6^1) + (5 \times 1)$.

6.

	Base Ten	Base Six	Base Five	Base Four	Base Three
(a)	11	15	21	23	102
(b)	15	23	30	33	120
(c)	28	44	103	130	1001
(d)	36	100	121	210	1100

7. Two. The binary base. With only one symbol it would be impossible to express both zero and one.

*8. (a) 1003_{four}
   (b) 1110_{six}
   (c) 1002_{three}
   (d) 424_{five}
   (e) 111_{four}
   (f) 1045_{six}
   (g) 112_{six}
   (h) 22_{three}

9. The new system is in base four.

Base ten	New Base	New Base Names
0	O	do
1	∣	re
2	∧	mi
3	≥	fa
4	∣O	re do
5	∣∣	re re
6	∣∧	re mi
7	∣≥	re fa
8	∧O	mi do
9	∧∣	mi re
10	∧∧	mi mi

Base ten	New Base	New Base Names
11	∧≥	mi fa
12	≥O	fa do
13	≥∣	fa re
14	≥∧	fa mi
15	≥≥	fa fa
16	∣OO	re do do
17	∣O∣	re do re
18	∣O∧	re do mi
19	∣O≥	re do fa
20	∣∣O	re re do

10.

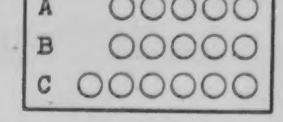
+	0	1	∧	≥
0	0	1	∧	≥
1	1	∧	≥	10
∧	∧	≥	10	11
≥	≥	10	11	1∧

×	0	1	∧	≥
0	0	0	0	0
1	0	1	∧	≥
∧	0	∧	10	1∧
≥	0	≥	1∧	∧1

2-8. The Binary and Duodecimal Systems.

Page 56. Junior high school pupils are often intrigued by a more extended discussion of computers than is found in the text. This is something you might bring out in classroom discussion. For example, you might say, "Now that we have seen how a row of lights may be used to represent a number using the binary system of numeration, how might we design a simple adding machine?" It is unimportant and quite beside the point, at this stage, for the pupil to be concerned with the electrical problems of design. He can, however, discover what such a machine might look like and what it must do. In the process he may gain some understanding of the fact that the binary system of numeration enables one to reduce computational problems to purely mechanical and electrical ones in a rather simple way.

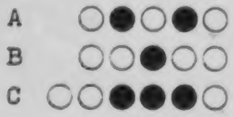
Suppose we wish to design a machine to add any two numbers which can be written in the binary scale with no more than 5 digits. We will need three rows of lights.



The numbers to be added will be entered into the machine by turning on appropriate lights in rows A and B. The sum is to be shown in Row C. The switches which operate the lights in Row C are to be wired in such a way that they are activated by the lights in rows A and B. Our task is to describe just when a light in Row C is to be on and when it is to be off.

Consider a few simple addition problems, and observe how they appear both in binary numerals and in lights. Suppose we wish to add 1010_{two} and 100_{two}.

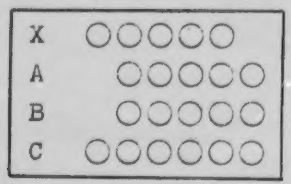
$$\begin{array}{r} 1010_{\text{two}} \\ 100_{\text{two}} \\ \hline 1110_{\text{two}} \end{array}$$



From this problem we can recognize two requirements for the wiring of our machine.

1. In a given column, if the lights in both Rows A and B are off then the light in that column in Row C should also be off.
2. In a given column, if one of the two lights in Rows A and B is on then the light in that column in Row C should turn on.

Now consider what happens when we add 1011_{two} and 101_{two}. Here we need to "carry," and our machine is not equipped to do this. It doesn't know what to do if two lights in a given column are on. We need one more row of lights; a "carrying row." The machine must look like this:



with Row X used only for carrying. The second addition looks like this:

$$\begin{array}{r} 111\\ 1011_{two}\\ \underline{101_{two}}\\ 10000_{two}\end{array}$$

X	○	○	●	●	●	
A		○	●	○	●	●
B		○	○	●	○	●
C	○	●	○	○	○	○

When we turn on lights in Rows A and B as shown, the indicated lights in Rows C and X must turn on automatically. Thus we see a third requirement for our wiring.

3. If, in a given column, any two lights in Rows A, B, or X are on then the light in Row C will remain off and the light in Row X in the column immediately to the left will turn on.

Requirement 3 says that if we have

X	○
A	●
B	●

or

X	●
A	○
B	●

then we must have

X	●	○
A		●
B		●
C		○

or

X	●	●
A		○
B		●
C		○

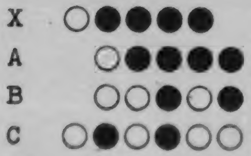
Finally, we have

4. If, in a given column, all the lights in Rows A, B, and X are on, then the light in Row C must go on and also the light in Row X of the column immediately to the left is to go on.

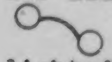
The addition

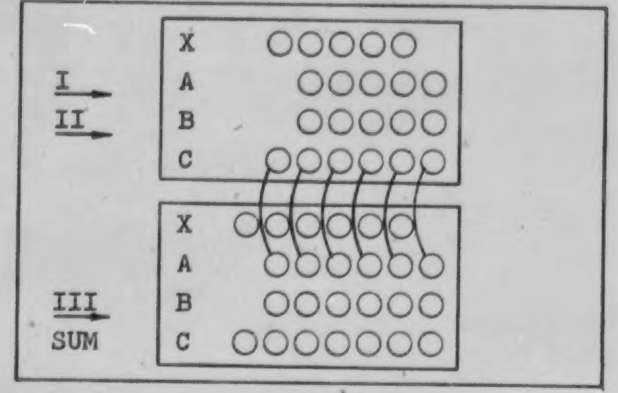
$$\begin{array}{r} 1111_{two}\\ \underline{101_{two}}\\ 10100_{two}\end{array}$$

would appear on the machine as



and the sum 10100_{two} would appear in Row C of lights almost as soon as we had entered the numbers 1111_{two} and 101_{two} in Rows A and B. We have designed an adding machine in the sense that requirements 1, 2, 3 and 4 can be built into a box with the required rows of lights by means of switches and relays. The important thing for the pupil at this stage is not the electrical problem, itself, but the knowledge that the binary system of notation makes it possible to reduce computational problems to purely mechanical and electrical construction in this rather simple way.

As an extension of the discussion, you might ask how we might use two of these adding machines to build a new machine for adding three numbers together. If we use the notation  to indicate two lights wired so that they are always lighted or unlighted together, then such a machine could be diagrammed as follows



If each of the smaller boxes within the larger box is an adding machine of the type described above then the large box is a computer which will add triples of numbers. If the binary numerals of the numbers are entered in Rows I, II and III, the sum appears in the bottom Row C.

The use of binary notation in high speed computers is, of course, well known. The binary system is used for computers since there are only two digits, and an electric mechanism is either "on" or "off." Such an arrangement is called a flip-flop mechanism. A number of pamphlets distributed by IBM, Remington Rand, and similar sources may be obtained by request and used for supplementary reading and study. "Yes No - One Zero" published by Esso Standard Oil Co., 15 West 51st., New York 19, New York is available for the asking only in states served by Esso.

It should be of interest that the sum 11001 + 110 looks the same in the binary system, decimal system, and, in fact, all positional number systems. The meaning, however, is quite different.

The base two has the disadvantage that, while only two different digits are used, many more places are needed to express numbers in binary notation than in decimal, e.g.,

$$2000_{ten} = 11,111,010,000_{two}$$

Pupils may be interested in the remainder method for changing a number from one base to another. This method of changing 25_{ten} to binary notation rests on repeated division by 2, to identify the powers of 2 whose sum is 25. To change 25_{ten} to base three, repeat division by 3; to base four, divide by 4; and so on.

The division is shown below for changing 25_{ten} to binary notation, followed by an interpretation of the results of the division at each stage. It will be noted that the remainders in reverse order indicate the digits in binary numerals. Recall that $2^0 = 1$.

$$\begin{array}{llr}
2\overline{)25} & & 25 = 25 \times 2^0 \\
2\overline{)12} & , R\ 1 & 25 = 12 \times 2^1 + 1 \times 2^0 \\
2\overline{)\ 6} & , R\ 0 & 25 = 6 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\
2\overline{)\ 3} & , R\ 0 & 25 = 3 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\
2\overline{)\ 1} & , R\ 1 & 25 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\
2\overline{)\ 0} & , R\ 1 & 25 = 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0
\end{array}$$

Here is a set of cards which can be used in a number trick.

1			
1	9	17	25
3	11	19	27
5	13	21	29
7	15	23	31

2			
2	10	18	26
3	11	19	27
6	14	22	30
7	15	23	31

4			
4	12	20	28
5	13	21	29
6	14	22	30
7	15	23	31

8			
8	12	24	28
9	13	25	29
10	14	26	30
11	15	27	31

16			
16	20	24	28
17	21	25	29
18	22	26	30
19	23	27	31

Using the first four cards, tell a person to choose a number between 1 and 15, to pick out the cards containing that number and to give them to you. By adding the numbers at the top of the cards he gives you, you can tell him the number he chose. Note that the numerals at the top of the cards represent the powers of two.

By using all five cards, you can pick out numerals from 1 to 31. The trick is based on the application of the binary numerals.

If you have a peg board and some match sticks, you can represent base two numerals on the board. Leave a hole blank for 0 and put in a match stick for one. Represent two numbers on the board, one below the other, and try adding on the board.

The twelve system uses two digits more than the decimal system. From some points of view twelve is a better choice for a base than ten. Many products are packaged and sold by the dozen and by the gross. Twelve is divisible by 2, 3, 4, and 6 as well as by 12 and 1. Ten is divisible only by 1, 2, 5, and 10. Because it employs a larger base, large numbers may be represented in base twelve with fewer digits than smaller bases require. For example:

$$\text{TOE}_{\text{twelve}} = 1451_{\text{ten}}$$

Answers to Exercises 2-8. Page 58.

1.

Base ten	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
Base two	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111	10000	10001	10010	10011	10100	10101	10110	10111	11000	11001	11010	11011	11100	11101	11110	11111	100000	100001

2. Addition, Base two

+	0	1
0	0	1
1	1	10

There are only four addition "facts."

3. Multiplication, Base Two

×	0	1
0	0	0
1	0	1

There are only four multiplication "facts." The two tables are not alike, except that 0 + 0 and 0 × 0 both equal 0.

The binary system is very simple because there are only four addition and four multiplication "facts" to remember. Computation is simple. Writing large numbers, however, is tedious.

4. (a) $111_{\text{two}} = (1 \times \text{two}^2) + (1 \times \text{two}) + (1 \times \text{one}) = 7$

(b) $1000_{\text{two}} = (1 \times \text{two}^3) + (0 \times \text{two}^2) + (0 \times \text{two}) + (0 \times \text{one}) = (1 \times 2^3) = 8$

(c) $10101_{\text{two}} = (1 \times \text{two}^4) + (0 \times \text{two}^3) + (1 \times \text{two}^2) + (0 \times \text{two}) + (1 \times \text{one}) = (1 \times 2^4) + (1 \times 2^2) + (1 \times 1) = 21$

[page 58]

(d) $11000_{\text{two}} = (1 \times \text{two}^4) + (1 \times \text{two}^3) + (0 \times \text{two}^2) + (0 \times \text{two}) + (0 \times \text{one}) = (1 \times 2^4) + (1 \times 2^3) = 24$

(e) $10100_{\text{two}} = (1 \times \text{two}^4) + (0 \times \text{two}^3) + (1 \times \text{two}^2) + (0 \times \text{two}) + (0 \times \text{one}) = (1 \times 2^4) + (1 \times 2^2) = 20.$

5. $22 = 1T_{\text{twelve}}$
   $23 = 1E_{\text{twelve}}$
   $24 = 20_{\text{twelve}}$

6. 408

7. (a) $111_{\text{twelve}} = (1 \times \text{twelve}^2) + (1 \times \text{twelve}) + (1 \times \text{one}) = (1 \times 144) + (1 \times 12) + (1 \times 1) = 157$
   (b) $3T2_{\text{twelve}} = (3 \times \text{twelve}^2) + (T \times \text{twelve}) + (2 \times \text{one}) = (3 \times 144) + (10 \times 12) + (2 \times 1) = 554$
   (c) $47E_{\text{twelve}} = (4 \times \text{twelve}^2) + (7 \times \text{twelve}) + (E \times \text{one}) = (4 \times 144) + (7 \times 12) + (11 \times 1) = 671$
   (d) $TOE_{\text{twelve}} = (T \times \text{twelve}^2) + (0 \times \text{twelve}) + (E \times \text{one}) = (10 \times 144) + (11 \times 1) = 1451$

8. (a) 111_{two} (c) 110000_{two}
   (b) 1011_{two} (d) 110110_{two}

9. (a) 10_{two} (c) 111_{two}
   (b) 11_{two} (d) 11_{two}

10. (a) 100011_{two} (c) 1100_{two}
    (b) 10000000_{two} (d) 1100100_{two}

11. (a) Addition 323_{twelve} (c) Subtraction 149_{twelve}
    (b) 1005_{twelve} (d) $85E_{\text{twelve}}$

[page 59]

12. The binary system is extremely simple in computation. Large numbers are tedious to write.

    The duodecimal system may be used conveniently to represent large numbers. Twelve is divisible by 1, 2, 3, 4, 6 and 12 while ten is divisible only by 1, 2, 5 and 10. The twelve system requires more computational "facts" which will increase difficulties in memorizing tables of addition and multiplication. We do use twelve in counting dozens, gross, etc., and in some of the common measures of length.

13. (a) $2E5_{twelve}$

    (b) 378_{twelve}

14. Five weights; 1 oz., 2 oz., 4 oz., 8 oz., may be used to check any weight up to 15 ounces. By adding a 16 oz. weight, any weight up to 31 ounces may be checked.

15. People who work with computers often use the base eight. To change from binary to octal and back is simple with the help of the table:

Binary	Octal
000	0
001	1
010	2
011	3
100	4
101	5
110	6
111	7

For example, we have

$$2000_{ten} = 011,111,010,000_{two} = 3720_{eight}$$

Note the grouping of numerals by threes in the binary numeral. The sum of the place values of digits in each group results in the octal numeral. Hence

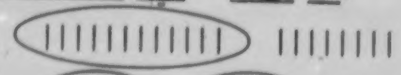
$$011 = (1 \times 2) + (1 \times 1) = 3$$

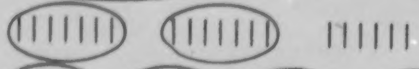
$$111 = (1 \times 4) + (1 \times 2) + (1 \times 1) = 7, \text{ etc.}$$

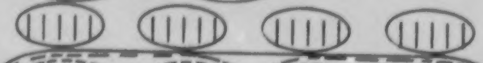
2-9. Summary.

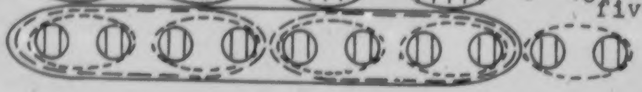
Page 61. The discussion is self-explanatory. Expansion on any of the topics is left to the discretion of the teacher.

Answers to Exercises 2-9. Page 62.

1. (a)  $= 18_{\text{twelve}}$

   (b)  $= 26_{\text{seven}}$

   (c)  $= 40_{\text{five}}$

   (d)  $= 10100_{\text{two}}$

2. (a) twelve (c) five
   (b) seven (d) two

3. (a) $111_{\text{two}} = (1 \times \text{two}^2) + (1 \times \text{two}) + (1 \times \text{one})$
   $= (1 \times 4) + (1 \times 2) + (1 \times 1) = 7$

   (b) $321_{\text{four}} = (3 \times \text{four}^2) + (2 \times \text{four}) + (1 \times \text{one})$
   $= (3 \times 16) + (2 \times 4) + (1 \times 1) = 57$

   (c) $2631_{\text{seven}} = (2 \times \text{seven}^3) + (6 \times \text{seven}^2) + (3 \times \text{seven})$
   $+ (1 \times \text{one}) = (2 \times 343) + (6 \times 49) +$
   $(3 \times 7) + (1 \times 1) = 1002$

   (d) $37T_{\text{twelve}} = (3 \times \text{twelve}^2) + (7 \times \text{twelve}) + (T \times \text{one})$
   $= (3 \times 144) + (7 \times 12) + (10 \times 1) = 526$

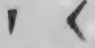
4. $1000_{\text{ten}} = 1750_{\text{eight}} = 1{,}111{,}101{,}000_{\text{two}}$

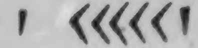
5. 100_{nine}, 101_{nine}, 102_{nine}, 103_{nine}, 104_{nine}.

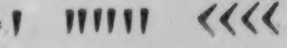
6. (a) 2; 61; 3600; 216,001; etc.

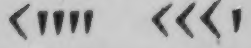
   (b) No. It cannot because the decimal system has a symbol for zero. If this symbol doesn't appear between the two 1's then no zero can be intended.

   (c) 𒐖 𒌋𒐖 𒌋𒁹 $= (𒐖 \times \text{sixty}^2) + (𒌋𒐖 \times \text{sixty}) +$
   $(𒌋𒁹 \times \text{one})$
   $= (2 \times 3600) + (12 \times 60) + (11 \times 1)$
   $= 7931$

(d) 70_{ten} = 

111_{ten} = 

4000_{ten} = 

(e) 

*7. The base is twenty. Four score and seven = 47_{twenty} = 87_{ten}.

*8. Since there are only five symbols the base is five.

DCBAO = $43{,}210_{five}$

$= (4 \times five^4) + (3 \times five^3) + (2 \times five^2) + (1 \times five) + (0 \times one)$

$= (4 \times 625) + (3 \times 125) + (2 \times 25) + (1 \times 5) + (0 \times 1)$

$= 2930_{ten}$.

9. 4

10. The base is twenty-six.

BE = $25_{twenty\text{-}six} = (2 \times twenty\text{-}six) + (5 \times one)$

$= (2 \times 26) + (5 \times 1) = 57_{ten}$

TWO = $(T \times twenty\text{-}six^2) + (W \times twenty\text{-}six) + (0 \times one)$

$= (19 \times twenty\text{-}six^2) + (22 \times twenty\text{-}six) + (0 \times one)$

$= (19 \times 676) + (22 \times 26) + (0 \times 1)$

$= 13{,}416$

FOUR = $(F \times twenty\text{-}six^3) + (0 \times twenty\text{-}six^2) + (U \times twenty\text{-}six) + (R \times one)$

$= (6 \times twenty\text{-}six^3) + (0 \times twenty\text{-}six^2) + (20 \times twenty\text{-}six) + (17 \times one)$

$= (6 \times 17{,}576) + (0 \times 676) + (20 \times 26) + (17 \times 1)$

$= 105{,}993_{ten}$

11. The method works for base twelve and base seven. It will also work for other bases. For bases larger than ten, add. For bases less than ten, subtract.
Example in Base Six:

$$44_{six} \quad \text{multiplying: } (4 \times 4) = 16$$

$$44 - 16 = 28_{ten}$$

Example in Base Fifteen:

$$46_{fifteen} \quad \text{multiplying: } (4 \times 5) = 20$$

$$46 + 20 = 66_{ten}$$

Students may suggest other methods which should be checked carefully for validity.

Sample Questions for Chapter 2

The sets of questions presented at the end of each chapter in the Teachers' Commentary are not intended as a chapter test. Teachers should construct chapter tests carefully by combining selected items from the set of questions included in this book and questions of their own writing. Great care should be used to avoid making the test too long.

Part I. True - False

1. The 3 in 356_{seven} stands for three hundred.
2. 10^4 means 10 × 10 × 10 × 10.
3. The numeral 8 means the same number in the ten system as in the twelve system.
4. The smaller the base, the more basic combinations there are in the multiplication table.
5. The fourth place from the right in the decimal system has the place value 10^5.
6. In base two numerals the number after 100 is 1000.
7. We can make a symbol to mean what we wish.
8. When we "carry" or "regroup" in addition the value of what is carried depends upon the base.

[pages 64-65]

9. A number may be written in numerals with any whole number greater than one as a base.
10. In the symbol 6^3, the exponent is 3.
11. 513_{six} means (5 × six × six × six) + (1 × six × six) + (3 × six).
12. The 1 in 10,000 (base two) means 1×2^4 or sixteen.
13. The following numerals represent the same number: 183_{twelve}; 363_{eight}; 1033_{six}
14. In base eight numerals, the number before 70 is 66.
15. Four symbols are sufficient for a numeration system with base five.
16. In the base four system $3 + 3 = 11_{\text{four}}$.
17. When we "borrow" in the twelve system as in 157 - 6E, we actually "borrow" twelve units.
18. In the Egyptian system a single symbol could be used to represent a collection of several things.
19. The Babylonians made use of place value in their numeration system.
20. The Roman numeral system had a symbol for zero.

Part II. Completion

1. In decimal numerals 14_{twelve} is ______.
2. MCXXIV in decimal numerals is ______.
3. The decimal system uses ______ different symbols.
4. In any numeration system, the smallest place value for whole numbers is ______.
5. 629,468,000,000 written in words is __________________.
6. The number represented by 212_{seven} is _________ (even or odd).
7. In expanded notation $5{,}678_{\text{ten}}$ is ________.
8. $625_{\text{seven}} + 344_{\text{seven}} =$ ______________$_{\text{seven}}$.
9. Multiply 312_{four} by 32_{four}.
10. $110011_{\text{two}} =$ _________$_{\text{ten}}$.
11. The numeral 444_{five} represents an _______ (even, odd number).
12. Add: $42_{\text{five}} + 14_{\text{five}} =$ _________.

13. 13_{ten} = ________$_{two}$.
14. The numeral after 37_{eight} is ________$_{eight}$.
15. Using all of the digits 5, 6, 7, and 0, write the largest possible 4-place base ten numeral ________.
16. Using all of the digits, 5, 6, 7, and 0, write the smallest possible 4-place base ten numeral________.
17. What is the largest number you can write, using two 4's and no other symbols?________.
18. Write this numeral without exponents: 5^3 ________.
19. The numeral immediately before 1000_{two} is ________.
20. Subtract: $42_{five} - 14_{five}$ = ________.

Part III. Multiple-Choice

1. In which of the numerals below does 1 stand for four?
   (a) 21_{four}
   (b) 21_{eight}
   (c) 100_{two}
   (d) 102_{three}
   (e) None of the above is correct.
2. In what base are the numerals written if $2 \times 2 = 10$?
   (a) Base two
   (b) Base three
   (c) Base four
   (d) Base five
   (e) All of the above are correct.
3. A decimal numeral which represents an odd number is:
   (a) 461,000
   (b) 7629
   (c) 5634
   (d) 9,000,000
   (e) None of the above is correct.
4. If N represents an even number, the next consecutive even number can be represented by:
   (a) N

(b) $N + 1$
(c) $N + 2$
(d) $2N$
(e) All of the above are correct.

5. Which numeral represents the largest number?
(a) 43_{five}
(b) 212_{three}
(c) 10110_{two}
(d) 24_{nine}
(e) $10_{twenty\text{-}five}$

6. Which is correct?
(a) $5^4 = 5 + 5 + 5 + 5$
(b) $4^3 = 4 \times 4 \times 4$
(c) $5^4 = 4 \times 4 \times 4 \times 4 \times 4$
(d) $2^3 = 2 \times 3$
(e) None of the above is correct.

7. 6120_{nine} is how many times as large as 612_{nine}?
(a) twelve
(b) ten
(c) nine
(d) five
(e) None of the above is correct.

8. In which base does the numeral 53 represent an even number?
(a) twelve
(b) ten
(c) eight
(d) seven
(e) six

Answers to Sample Questions for Chapter 2

Part I. True - False

1. False
2. True
3. True
4. False
5. False
6. False
7. True
8. True
9. True
10. True
11. False
12. True
13. False
14. False
15. False
16. False
17. True
18. True
19. True
20. False

Part II. Completion

1. 16
2. 1124
3. Ten
4. One
5. Six hundred twenty-nine billion, four hundred sixty-eight million.
6. Odd
7. $(5 \times 10^3) + (6 \times 10^2) + (7 \times 10^1) + (8 \times 1)$
8. 1302_{seven}
9. 23310_{four}
10. 51
11. even
12. 111_{five}
13. 1101_{two}
14. 40_{eight}
15. 7650
16. 5067
17. 4^4
18. 125
19. 111_{two}
20. 23_{five}

Part III. Multiple Choice

1. (c)
2. (c)
3. (b)
4. (c)
5. (e)
6. (b)
7. (c)
8. (d)

Chapter 3

WHOLE NUMBERS

3-1. Counting Numbers.

Page 67. Understandings to be developed in this section:

(a) Number is an idea.

  (1) Small numbers can be learned without counting, some large ones by patterns. (Cards, Dominoes)

  (2) The number of members of any set can be found by matching the members of the set with some standard set. This is a clumsy method if the number of members is large, as the standard sets must themselves be large.

  (3) Matching the members of a set with a memorized set of sounds representing the counting numbers is the best known way of finding the number of members in a set.

(b) One-to-one correspondence.

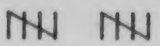
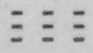
  Two sets are in a one-to-one correspondence when each member of one set is matched with only one member of the other set and vice-versa.

(c) "Counting numbers" are sometimes called "natural numbers"

(d) The name that we give to the set of counting numbers and zero is "whole numbers." The counting numbers do not include zero.

Answers to Exercises 3-1. Pages 68-70.

1. (a) 1, 2, 3, 4, 5, 6.
   (b) 1 + 1, 2 + 1, 3 + 1, 0 + 5, 5 + 1, 1 + 6, 5 + 3.
   (c) IV, V, VI, VII, VIII, IX, X, XI.
   (d) 1_{seven}, 2_{seven}, 3_{seven}, 4_{seven}, 5_{seven}, 6_{seven}, 10_{seven}.
2. all; 7 and 8.
3. Putting them in a 5 × 5 square will probably be as simple a pattern as any other.

4. (a) 
   (b) 
5. Multiply 4 by 3; (2 × 5) + (2 × 1); 2 × 6 on each side of diagonal; 2 × 6 on each side of the vertical line segment; any other way that doesn't use one-to-one correspondence.
6. French - un; Spanish - unos; German - ein; Russian - **adeen**
7. IV corresponds to two dots.
8. 8 was left out.
9. 17
10. He subtracted 27 from 81. No, the answer should be 55.
11.

Two-to-one correspondence	Four-to-one correspondence
(a) Feet to people	(a) Tires to cars, wagons
(b) Ears, eyes, arms, hands etc., to people.	(b) Legs to dogs, horses cows

12. The following illustrates a one-to-one correspondence between the counting numbers and the even numbers.

3-2. Commutative Properties for Whole Numbers.

Page 70. The principal objective in the study of the commutative, associative, and distributive properties is to have the pupils understand their statements in mathematical language; to distinguish one property from another; and to recognize the property or properties, that may be used in such problems as in Exercises 3 and 4. These are not properties that are being proved. The pupils have used them for a long time, but they probably have not had names for them and have not recognized when they have been using them.

The pupils will be helped in understanding the commutative property if they have some small objects with which to work. Pieces of paper will do but cubical blocks will be better. Cubical blocks will be more useful in illustrating the associative property for multiplication. Emphasize that the commutative property is a property of (belongs to) the operation of addition and is not dependent upon the numeration system that is used.

Use Roman numerals and numbers in base five to illustrate. For example in base five add

$$12_{five} + 23_{five} = 23_{five} + 12_{five} = 40_{five}$$

$$\underbrace{\text{*****}\quad\text{**}}_{12}\qquad\underbrace{\text{*****}\quad\text{*****}\quad\text{***}}_{23}\qquad\underbrace{\text{*****}\quad\text{*****}\quad\text{*****}\quad\text{*****}}_{40}$$

This is a good opportunity to review numbers in numeration systems other than that with base ten.

Students should be taught to add "up the column" as well as "down the column." Use three whole numbers in the addition since the use of three numbers will not only illustrate commutativity but will lead very naturally into associativity.

$$\begin{array}{r} 476 \\ 324 \\ \underline{935} \end{array}\Big\downarrow \qquad\qquad \begin{array}{r} \overline{476} \\ 324 \\ 935 \end{array}\Big\uparrow$$

Commutative Property of Multiplication

Diagrams of stars on the blackboard, arrangements of the blocks in rows and columns will be helpful.

*****	
*****	Five rows of 4 stars in each row
*****	Four columns of 5 stars in each column
*****	

After repeated arrangements of stars (or blocks) in rows and columns strive for understanding of $a \cdot b = b \cdot a$ where a and b represent any whole numbers.

Answers to Exercises 3-2a. Page 73.

1. Parts a, c, d, f, h, i, m are true
   Parts b, e, g, j, k, l are false
2. (a) 644 (b) 110,596 (c) 155,752 (d) 105_{seven}
3. (a) $7 + 4 = 4 + 7$
   (b) $12 \cdot 5 > 5 \cdot 11$
   (c) $23 \cdot 12 < 12 \cdot 32$
   (d) $3 < 6$
   (e) $16 > 9 > 3$
   (f) $(3 \cdot 2) + 5 = 5 + (3 \cdot 2)$
   (g) $8 - 3 < 9 - 3$
   (h) $86 \cdot 135 = 135 \cdot 86$
   (i) $24 \div 3 > 3 \div 24$
   (j) $a > c$
4. (a) 2052 (b) 25,620 (c) 289,884 (d) 1135_{seven}

5. (a) 5
   (b) 5
   (c) 0
   (d) 0, 1
   (e) 46
   (f) 0, 1, 2, 3, 4, 5, 6
   (g) 0, 1, 2, 3, 4
   (h) any whole number

Answers to Exercises 3-2b. Page 74.

1. Result is unchanged in parts a, b, c.
   Result is changed in parts d, e, f, g, h, i.
2. No.
3. The activities are commutative in parts a, c, e.
4. Yes.
5. The operation in part d is commutative. (The operations in parts a, b, c are commutative only if the first number in each is equal to the second.)
6. Examples of commutative activities.
   To wash your face and wash your hair.
   To go north one block and then west one block.
   To count to 100 and write the alphabet.
   Examples of activities which are not commutative.
   To put out the cat and go to bed.
   To eat dinner and get up from the table.
   To rake the leaves and burn them.

3-3. Associative Properties for Whole Numbers.

Page 74. Have the students use blocks or disks to make such arrangements as

*** ** ****

Then have them push the first two sets together and count the total $(3 + 2) + 4$. Then after rearranging have them push the second two sets together and count the total $3 + (2 + 4)$. Use sufficient variations of this procedure to lead to the understanding that $(a + b) + c = a + (b + c)$ where a, b, c are any counting numbers.

Then ask: Is the product $(3 \times 4) \times 5$ equal to the product $3 \times (4 \times 5)$?

This may be illustrated by arranging a set of 20 blocks in a rectangular array, 4 by 5. Then put two more layers of 20 blocks each on top of these forming a box arrangement. Look at it in different ways to see $(3 \times 4) \times 5$ and $3 \times (4 \times 5)$. Different sets of boxes may be made to illustrate $2 \times (3 \times 4)$, $(2 \times 3) \times 4$ and many others. Again, emphasis is upon arrival at understanding that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ where a, b, c are any whole numbers.

Point out some operations or activities which are not associative and have students suggest others.

Answers to Exercises 3-3. Pages 77-78.

1. (a) $(4 + 7) + 2 = 4 + (7 + 2)$ Associative property of addition
$(4 + 7) + 2 = 11 + 2 = 13$
$4 + (7 + 2) = 4 + 9 = 13$

(b) $8 + (6 + 3) = (8 + 6) + 3$ Associative property of addition
$8 + (6 + 3) = 8 + 9 = 17$
$(8 + 6) + 3 = 14 + 3 = 17$

(c) $46 + (73 + 98) = (46 + 73) + 98$ Associative property of addition
$46 + (73 + 98) = 46 + 171 = 217$
$(46 + 73) + 98 = 119 + 98 = 217$

(d) $(6 \cdot 5) \cdot 9 = 6 \cdot (5 \cdot 9)$ Associative property of multiplication
$(6 \cdot 5) \cdot 9 = 30 \cdot 9 = 270$
$6 \cdot (5 \cdot 9) = 6 \cdot 45 = 270$

(e) $(21 + 5) + 4 = 21 + (5 + 4)$ Associative property of addition
$(21 + 5) + 4 = 26 + 4 = 30$
$21 + (5 + 4) = 21 + 9 = 30$

(f) $(9 \cdot 7) \cdot 8 = 9 \cdot (7 \cdot 8)$ Associative property of multiplication
$(9 \cdot 7) \cdot 8 = 63 \cdot 8 = 504$
$9 \cdot (7 \cdot 8) = 9 \cdot 56 = 504$

(g) $436 + (476 + 1) = (436 + 476) + 1$ Associative property of addition
$436 + (476 + 1) = 436 + 477 = 913$
$(436 + 476) + 1 = 912 + 1 = 913$

(h) $(57 \cdot 80) \cdot 75 = 57 \cdot (80 \cdot 75)$ Associative property of multiplication

$(57 \cdot 80) \cdot 75 = (4560) \cdot 75$

$57 \cdot (80 \cdot 75) = 57 \cdot 6000 = 342{,}000$

2. (a) No. (b) No. (c) There is no associative property of subtraction, or the associative property of subtraction does not hold.

3. (a) No.

(b) No.

(c) $(75 \div 15) \div 5 = 1$

(d) $75 \div (15 \div 5) = 25$

(e) $80 \div (20 \div 2) = 8$

(f) $(80 \div 20) \div 2 = 2$

(g) The associative property does not hold for division.

4. (a) Either $(6 + 1) + 9 = 7 + 9 = 16$, or

$$\begin{aligned}(6 + 1) + 9 &= 6 + (1 + 9)\\ &= 6 + 10\\ &= 16\end{aligned}$$

Associative property of addition

(b) $2 \cdot (13 \cdot 10) = 2 \cdot 130 = 260$ Pupils may prefer this way.

(c) $(12 \cdot 9) \cdot 10 = (108) \cdot 10 = 1080$

(d)

$$\begin{aligned}4 \cdot (25 \cdot 76) &= (4 \cdot 25) \cdot 76\\ &= 100 \cdot 76\\ &= 7600\end{aligned}$$

Associative property of multiplication

(e)

$$\begin{aligned}340 + (522 + 60) &= 340 + (60 + 522)\\ &= (340 + 60) + 522\\ &= 400 + 522\\ &= 922\end{aligned}$$

Commutative property of addition.

Associative property of addition

(f)

$$\begin{aligned}(5 \cdot 67) \cdot 2 &= (67 \cdot 5) \cdot 2\\ &= 67 \cdot (5 \cdot 2)\\ &= 67 \cdot 10\\ &= 670\end{aligned}$$

Commutative property of multiplication.

Associative property of multiplication

3-4. The Distributive Property.

Page 79. Emphasize that the use of two operations in this property makes it easy to distinguish the distributive from the two preceding properties.

The blocks can be used in the following way. Lay out 2 rows of 3 each and 2 rows of 5 each.

***   *****
***   *****

Ask: If we move these together, we will have 2 times what? When we move them together do we get 2 times 8? This can be repeated until pupils understand that

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

when a, b, c are whole numbers. Repetition of the same illustration with different numbers of blocks may be better than different types of illustrations. For bright students, the distributive property can be extended to 3 terms. This can be illustrated and used with 3 digit numbers. The general form is:

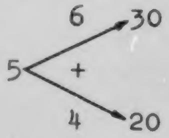
$$a(b + c + d) = ab + ac + ad$$

Use the distributive property to help make mental computations such as:

$$7 \cdot 32 = 7 \cdot (30 + 2) = (7 \cdot 30) + (7 \cdot 2) = 210 + 14 = 224$$
$$35 \cdot 8 = (30 + 5) \cdot 8 = (30 \cdot 8) + (5 \cdot 8) = 240 + 40 = 280$$
$$5 \cdot 152 = 5 \cdot (100 + 50 + 2) = (5 \cdot 100) + (5 \cdot 50) + (5 \cdot 2)$$
$$= 500 + 250 + 10 = 760$$

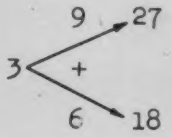
Answers to Exercises 3-4. Pages 82-84.

1. (a)



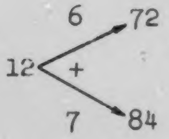
$= 30 + 20 = 50$

(b)



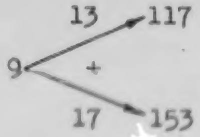
$= 27 + 18 = 45$

(c)



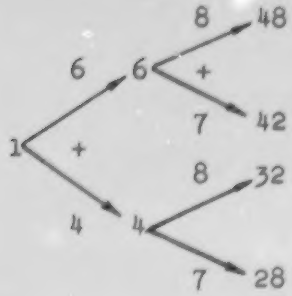
$= 72 + 84 = 156$

(d)


13
117
9
+
17
153

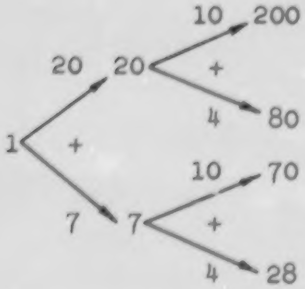
$= 117 + 153 = 270$

(e)


8
48
6
6
+
7
42
1
+
8
32
4
4
7
28

$= (48 + 42) + (32 + 28)$
$= 90 + 60$
$= 150$

(f)


10
200
20
20
+
4
80
1
+
10
70
7
7
+
4
28

$= (200 + 80) + (70 + 28)$
$= 280 + 98$
$= 378$

2. (a) $4\cdot 12 = 48$
$28 + 20 = 48$

(b) $18 + 24 = 42$
$6\cdot 7 = 42$

(c) $48 + 42 = 90$
$15\cdot 6 = 90$

(d) $23\cdot 5 = 115$
$46 + 69 = 115$

(e) $11\cdot 7 = 77$
$33 + 44 = 77$

(f) $30 + 18 = 48$
$6\cdot 8 = 48$

(g) $2\cdot 24 = 48$
$32 + 16 = 48$

(h) $12\cdot 5\frac{1}{4} = 63$
$60 + 3 = 63$

(i) $3216 + 3484 = 6700$
$67\cdot 100 = 6700$

(j) $36 + 36 = 72$
$72\cdot 1 = 72$

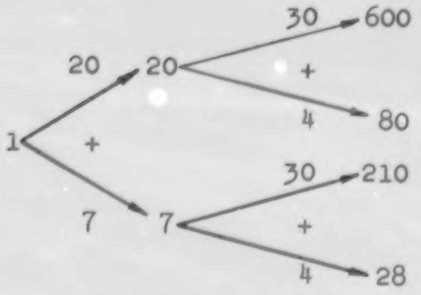
3. (a) $3\cdot(4 + \underline{3}) = (3\cdot 4) + (3\cdot 3)$
(b) $2\cdot(\underline{4} + 5) = (2\cdot 4) + (2\cdot 5)$
(c) $13\cdot(6 + 4) = (13\cdot\underline{6}) + (13\cdot\underline{4})$
(d) $(2\cdot 7) + (3\cdot 7) = (\underline{2} + \underline{3})\cdot 7$
(e) $(\underline{6}\cdot 4) + (\underline{7}\cdot 4) = (6 + 7)\cdot\underline{4}$

4. (a) $9\cdot(8 + 2)$ (d) $(13\cdot6) + (27\cdot6)$
   (b) $(8\cdot14) + (8\cdot17)$ (e) $(15\cdot6) + (15\cdot13)$
   (c) $(12\cdot5) + (12\cdot7)$ (f) $12\cdot(5 + 4)$

5. (a) $(5\cdot7) + (5\cdot8) = 5\cdot(7 + 8)$
   (b) $(3\cdot4) + (3\cdot5) = 3\cdot(4 + 5)$
   (c) $(5\cdot11) + (5\cdot2) = 5\cdot(11 + 2)$
   (d) $(3\cdot9) + (3\cdot17) = 3\cdot(9 + 17)$
   (e) $(5\cdot20) + (5\cdot23) = 5\cdot(20 + 23)$
   (f) $(3\cdot10) + (3\cdot7) = 3\cdot(10 + 7)$

6. The following parts are true: b, c, d.
   The following parts are false: a, e.

7. (a) $(20 + 7)\cdot(30 + 4) = 20\cdot(30 + 4) + 7\cdot(30 + 4)$
   $= (20\cdot30) + (20\cdot4) + (7\cdot30) + (7\cdot4)$
   $= 600 + 80 + 210 + 28$
   $= 918$

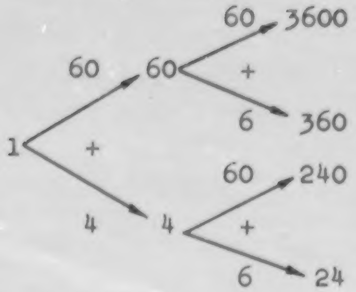


$= (600 + 80) + (210 + 28) = 918$

   (b) $(10 + 3)\cdot(20 + 2) = 10\cdot(20 + 2) + 3\cdot(20 + 2)$
   $=(10\cdot20) + (10\cdot2) + (3\cdot20) + (3\cdot2)$
   $= 200 + 20 + 60 + 6$
   $= 286$

   (c) $(30 + 7)\cdot(30 + 3) = 30\cdot(30 + 3) + 7\cdot(30 + 3)$
   $= (30\cdot30) + (30\cdot3) + (7\cdot30) + (7\cdot3)$
   $= 900 + 90 + 210 + 21$
   $= 1221$

(d) $$\begin{aligned}(60 + 4)\cdot(60 + 6) &= 60\cdot(60 + 6) + 4\cdot(60 + 6)\\ &= (60\cdot 60) + (60\cdot 6) + (4\cdot 60) + (4\cdot 6)\\ &= 3600 + 360 + 240 + 24\\ &= 4224\end{aligned}$$



$$= (3600 + 360) + (240 + 24) = 4224$$

(e) $$\begin{aligned}(70 + 5)\cdot(70 + 5) &= 70\cdot(70 + 5) + 5\cdot(70 + 5)\\ &= (70\cdot 70) + (70\cdot 5) + (5\cdot 70) + (5\cdot 5)\\ &= 4900 + 350 + 350 + 25\\ &= 5625\end{aligned}$$

(f) $$\begin{aligned}(20 + 1)\cdot(20 + 9) &= 20\cdot(20 + 9) + 1\cdot(20 + 9)\\ &= (20\cdot 20) + (20\cdot 9) + (1\cdot 20) + (1\cdot 9)\\ &= 400 + 180 + 20 + 9\\ &= 609\end{aligned}$$

8. From (a) to (b) Commutative property of multiplication
From (b) to (c) Associative property of multiplication
From (c) to (d) Commutative property of addition
From (d) to (e) Associative property of addition
From (e) to (f) Commutative property of multiplication
From (f) to (g) Associative property of multiplication
From (g) to (h) Distributive property

3-5. Sets and the Closure Property.

Page 84. Emphasis here is placed upon (1) understanding the meaning of set and (2) the meaning of a set closed under an operation. Pupils will give many examples of sets. There is just one empty set. The set of marbles which has no marbles in it is the same as the set of coins which a pupil has in his pocket if he has no coins in his pocket.

Good examples of sets closed under addition:

The set of whole numbers.

The set of counting numbers.

Then ask if they are closed under multiplication. Under subtraction? Under division?

Emphasize that if just one pair of counting numbers can be found such that their difference (or quotient) is not a counting number, then the set of counting numbers is not closed under subtraction (or division). For example, 9-12 is not a counting number for there is no counting number which can be added to 12 to get 9 and $\frac{12}{9}$ is not a counting number since there is no counting number which can be multiplied by 9 to get 12.

Answers to Exercises 3-5a. Pages 87-88.

1. (a) No. The sum of 2 odd numbers is always an even number.
   (b) No.
2. Yes, since the sum of 2 even numbers is always an even number.
3. Yes. Since each of the numbers in the set is a multiple of 5, the sum of any two numbers in the set is a multiple of 5.
4. Each set is closed under multiplication.
5. (a) Yes.
   (b) No. For example 500 + 501 = 1001 and 1001 is not in the set.
   (c) No. For example, 3 + 47 = 50, and 50 is not in the set.
   (d) Yes. If the numerals of 2 numbers end in 0, then the sum of the numbers ends in 0.
6. (a) Yes. (b) No. (c) No. (d) Yes.
7. Yes. Multiplication is an abbreviated process for addition.
8. No.
9. No.

Answers to Exercises 3-5b. Page 87.

1. (a) 8578 (b) 3165
2. 19997
3. 221312
4. 407
5. $27\frac{3}{4}$
6. (a) 4596 (b) 1470 (c) 5081
7. $20.10
8. $389.37
9. 4800
10. Two million, seventy thousand, three hundred fifty-one.
11. 72 cents
12. (a) 78,528 (b) 450,954 (c) 2,499,574

3-6. Inverse Operations.

Page 88. The basic concepts in this section are:

1. The meaning of inverse.
2. Addition and multiplication have the closure, commutative, and associative properties, while their inverses do not.

The meaning of inverse may be explained by giving an example. "I write on the chalkboard" may be stated as one actually writes "inverse" on the board. Then the teacher may say "The inverse of writing on the board is erasing the board." The board may be actually erased. If the phrase "opposite or inverse" is used it may help the pupil understand the meaning of inverse. It should be emphasized that the inverse operation undoes the first operation. Some pupils may think that the failure to do an operation is the inverse of the operation. For example, to the question "What is the inverse of singing?" the pupil may say "Not singing." "Not singing" is the negation of "singing" and is not the inverse of "singing." In this connection it is important to point out that some operations have no inverse.

Some discussion of $a \cdot x = b$ may be helpful to many students. The following questions may be suggestive.

1. What operation is indicated by $a \cdot x$?
2. What operation will undo multiplication?

3. What is the inverse of multiplication?
4. To undo $a \cdot x$, do we divide $a \cdot x$ by a or $a \cdot x$ by x? (Since $a \cdot x$ means a times x we divide by a, the multiplier.)
5. How do we undo $3 \cdot 2$? 8×4? Divide 6 by 3; divide 32 by 8?
6. In terms of these symbols, can you define division?

An understanding of $a \cdot x = b$ will be helpful to the pupil as he studies percentage, and the equivalence of the two statements "$b \div a = x$" and "$a \cdot x = b$" will be of great importance in Chapter 6. Therefore, an emphasis on understanding the relationship between a, x, and b is desirable.

Answers to Exercises 3-6. Pages 90-91.

1. The operations in the following parts have inverses: a, b, c, d, e, f, h, j, l, o.
2. (a) Put down the pencil.
   (b) Take off your hat.
   (c) Get out of a car.
   (d) Withdraw your hand.
   (e) Divide.
   (f) Tear down.
   (h) Step backward.
   (j) Subtraction.
   (l) Addition.
   (o) Putting on a tire.
3. (a) 46471
   (b) \$507.10
   (c) 506 ft.
   (d) \$1412.78
   (e) \$1101.04
   (f) \$1342.67
   (g) 876
   (h) 987
   (i) 798
   (j) 697
   (k) 33
   (l) 1476
   (m) 68
   (n) 143
   (o) 58140
4. (a) 5
   (b) 5
   (c) 1
   (d) 7
   (e) None
   (f) 0
   (g) 8
   (h) 4
   (i) 3
   (j) 3

(k) None
(l) 7
(m) Any whole number
(n) 4
(o) 5
(p) 5
(q) 9
(r) 9
(s) 6
(t) 0
(u) 0
(v) 0
(w) 0
(x) 1
(y) 1
(z) 1

5. (a) 19
(b) 1992
(c) 89
(d) 19,219
(e) 165821
(f) 13
(g) 6
(h) 20

6. (a) 21
(b) 84
(c) 202
(d) 3
(e) 46
(f) 20
(g) 104
(h) 5
(i) 11
(j) 11

3-7. Betweenness and the Number Line.

Page 93. Understandings

1. The number line helps to show how the counting numbers are related. The students may ask about the dots to the left of zero. These dots will be discussed next year.

2. A number is less than a second number if the first is to the left of the second. A number is greater than another if it is to the right of it.

3. There is not always a counting number between two counting numbers.

4. To find the number of whole numbers between two other numbers (if it can be done at all): Subtract the smaller from the larger and then subtract one (1) from the remainder. Or, subtract one (1) from the larger and then find the difference between that and the smaller. Or, add 1 to the smaller and then subtract from the larger. Example: Between 7 and 15.
   Method 1 $15 - 7 = 8$ $8 - 1 = 7$
   Method 2 $15 - 1 = 14$ $14 - 7 = 7$
   Method 3 $7 + 1 = 8$ $15 - 8 = 7$

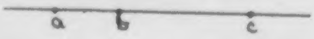
Answers to Exercises 3-7.

1. (a) 17
   (b) 21
   (c) 4
   (d) 7
   (e) None.
   (f) 2
   (g) None
   (h) 88
   (i) 3 is answer. Discuss with class that the answer is either $a - (b + 1)$ or $(a - b) - 1$ or $(a - 1) - b$.

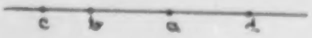
2. (a) 10
   (b) 11
   (c) 24
   (d) 30
   (e) 18
   (f) 22
   (g) 16
   (h) 9

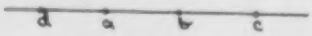
3. (a), (b), (c), (g), (i), (j).

4. (a) Yes. (b) Yes. (c) Yes.



5.





Either of the two situations is possible. The diagrams indicate that b is between c and d regardless of whether $c < d$ or $d < c$.

3-8. The Number One.

Page 94. In this lesson emphasis should be placed not only on special properties of the operations with 1 but also on the closure, associative and commutative properties. The fact that there is more than one way to represent the number 1 is emphasized in the first exercise. Of course this gives the teacher an opportunity to review the concept of numeral as a name for a number and not the number itself. Pupils think of the operation with numbers so frequently that they forget that $(4 - 3)$ is really another way to represent 1.

A class discussion of the lesson summary in symbols may be profitable for many pupils. Of course some other letter in place of C could be used as practice for pupils in translating symbols into words. Be sure that the pupils have the ideas before attempting symbolism. Pupils' translations of the mathematical sentences could be somewhat as follows:

(a) The counting numbers (C) are obtained by starting with 1 and repeatedly adding 1.

(b) If any counting number is multiplied by 1, the product is the same number.

(c) If any counting number is divided by 1, the quotient is the same counting number.

(d) If any counting number is divided by the same number, the quotient is 1.

(e) The number one, raised to any power which is a counting number, equals 1.

Answers to Exercises 3-8. Page 96.

1. The symbols in the following parts represent the number 1:
   (a), (b), (c), (d), (e), (i), (k), (l), (m), (o), (p).

2. (a) $100 \cdot 1 = 100$ (d) $1 \cdot \frac{2}{3} \cdot 1 = \frac{2}{3}$
   (b) $10 \cdot 1 \cdot 1 \cdot 1 = 10$ (e) $0 \cdot 1 = 0$
   (c) $\frac{14}{1} = 14$ (f) $1 \cdot 0 = 0$

3. We can get any counting number by the repeated addition of 1 to another counting number if the number we wish to get is larger than the counting number to which we add.
   We can get any counting number by the repeated subtraction of 1 from another counting number if the number we wish to get is smaller than the counting number from which we subtract.

4. Yes. $1 - 1 = 0$; $3 - 1 - 1 - 1 = 0$. Zero is not a counting number.

5. The successive addition of 1's to any counting number will give a counting number. But, the successive subtraction of 1's from any counting number will become 0 if carried far enough.

6. (a) 876429 (c) 897638 (e) 3479 (g) 1
   (b) 976538 (d) 896758 (f) 97 (h) 1

3-9. The Number Zero.

Page 97. The purpose of this lesson is to understand why we can or cannot perform the fundamental operations with zero.

It is important for pupils to understand that zero is a perfectly good number and that it does not mean "nothing." The pupil should see that, in addition and subtraction, zero obeys the same laws as the counting numbers.

In explaining the product of $c \cdot 0$ and $0 \cdot c$, it may be helpful to review briefly the meaning of multiplication. Such discussion questions might be:

1. What is another way to find the answer to 3×5?
2. What does 3×5 mean? It means $5 + 5 + 5$ and not $3 + 3 + 3 + 3 + 3$.
3. Make up a real problem using 3×5. (The price of 3 pencils at 5¢ each.)
4. What does 5×3 mean? (It means $3 + 3 + 3 + 3 + 3$.)
5. Make up a real problem using 5×3. (The price of 5 pencils at 3¢ each.)

After such questions, zero may be introduced in the discussion as multiplicand and multiplier, since $5 \cdot 0 = 0 \cdot 5$ by the commutative property for multiplication.

In case of division, pupils should understand why we divide 0 by a and do not divide a by 0. It may be desirable to use several examples so that the pupils will see that $\frac{0}{c}$ should be 0 and $\frac{c}{0}$ is not the name of any whole number.

Some of the pupils may be interested in why we do not define $\frac{c}{0} = 1$ or some other number. They should understand that it would be out of harmony with the fact that zero times any number equals zero.

The translations into words of the symbolic statements concerning zero can be somewhat as follows:

(a) If zero is added to any whole number, the sum is the same whole number.

(b) If any whole number is added to zero, the sum is the same whole number.

(c) If zero is subtracted from any whole number, the difference is the same whole number.

(d) If any whole number is multiplied by zero, the product is zero.

(e) If zero is multiplied by any whole number, the product is zero.

(f) If the product of any two whole numbers is zero, then one of the whole numbers is zero or both are zero.

(g) If zero is divided by any counting number, the quotient is zero.

(h) Zero cannot be used as a divisor.

Answers to Exercises 3-9. Page 101.

1. The symbols in the following parts represent zero: (b), (d), (f), (h), (i), (j), (k), (l), (m), (n), (o), (q), (s).

2. (a) 18424
(b) 641388
(c) 144, remainder 56
(d) 152, remainder 60
(e) \$36538.26
(f) \$60477.81
(g) 0
(h) \$846.25
(i) \$70.65
(j) 679
(k) 379
(l) 897
(m) \$397.16
(n) Division by zero not possible
(o) 1
(p) \$1846
(q) 0
(r) 0
(s) 0
(t) 0
(u) 976
(v) \$97.46

3. The error is in the generalization to c in part (i). If $a \cdot b = c$, a or b does not need to be c. Example: $2 \cdot 2 = 4$. This exercise shows the error that may be made by making a generalization on a few cases.

3-10. Summary. Page 102. (In student text only)

Answers to "How Are You Doing?" Questions, Chapters 1-3.

1. $(122)_{\text{three}} = (17)_{\text{ten}} = (32)_{\text{five}}$ (It is easiest to get to base five by going through base ten.)
2. Yes. Start by filling either the 3-cup or the 5-cup container. If the three-cup is filled first then:
   (a) Pour 3 into 5; (b) Fill 3; (c) Pour 2 from 3 into 5, which leaves 1 in 3; (d) Empty 5, pour 1 left in 3 into 5; (e) Fill 3. Now we have 4.
   If the five-cup is filled first then:
   (a) Pour 3 from 5 into 3, which leaves 2 in 5; (b) Empty 3 and pour in 2 from 5; (c) Fill 5; (d) Fill 3 by pouring 1 from 5; (e) Empty 3. Now we have 4.
3. $\frac{1}{6}$
4. 1,111
5. 176 × 176 × 176 × 176 × 176 × 176 × 176
6. Addition
7. Base 2
8. $(2 \times 27) + (0 \times 9) + (1 \times 3) + (0 \times 1)$
9. 31
10. 4×216 is the value of the 4 in $(4512)_{\text{six}}$ which equals 864. 4×27 is the value of the 4 in $(41)_{\text{twenty-seven}}$, which equals 108. 864 is 8 times 108.
11. The set of whole numbers has a zero and the set of counting numbers has not.
12. The statement is false. If one can find one example where the operation is not closed, then this operation is not closed for the set of whole numbers.
13. Commutative property of multiplication.
14. One (1)
15. Any of the properties listed in the section on zero.
16. $13 \cdot (2 + 5)$ or $(2 + 5) \cdot 13$
17. $136 + (25 + 75)$
18. Multiply 65 times 11 and see if the product is 715.
19. 40
20. $(7 \cdot 3) + 6 \cdot (5 \cdot 3)$

Sample Questions for Chapter 3

The sets of questions presented at the end of each chapter in the Teachers' Commentary are not intended as a chapter test. Teachers should construct chapter tests carefully by combining selected items from the set of questions included in the guide and questions of their own writing. Great care should be used to avoid making the test too long.

1. Insert a symbol which makes a true statement:
   $$8 + 4 \quad\quad 4 + 8$$
2. How many days are there between March 13, 1951 and March 27, 1951?
3. Show with one example that the set of numbers from 10 to 15 is not closed under addition.
4. Answer true or false: The identity for multiplication in the set of whole numbers is zero.
5. If K is a counting number then $\frac{0}{K} = ?$
6. Apply the commutative property of addition to: $(4 \cdot 5) + 6$
7. We are using the ________________ property when we say that $3a + 5a$ is another way of writing $(3 + 5) \cdot a$.
8. If the product of 5 and a certain number is zero, then that number must be: (a) 1 (b) 0 (c) 5
   (d) None of the above
9. When the number one is divided by any counting number n, the answer is always: (a) 0 (b) 1 (c) n
   (d) None of the above
10. Which of the following numerals are names of counting numbers? $(10)_{two}$ 14 0 $(713)_{ten}$ $\frac{2}{3}$ XIV
11. $[(7 + 3) + 7 \cdot (6 \cdot 5)] = [(7 + 3) + (7 \cdot 6) \cdot 5]$ illustrates the associative property of ___?____
12. How many counting numbers are there between $(10)_{five}$ and $(13)_{four}$?
13. Make a true statement of $(3 \cdot 7) + (4 \quad) \quad 7 \cdot (\quad\quad)$
14. To check the statement $7 \times 345 = 2415$ by the inverse operation we would ____?____.

15. The letters a, b, and x represent counting numbers, and $\frac{a}{b} = x$. What can we say about the relation between a and b?

Answers to Sample Questions

1. =
2. 13
3. $12 + 13 = 25$
4. False
5. 0
6. $6 + (4 \cdot 5)$
7. Distributive
8. b
9. a
10. $(10)_{two}$, 14, $(713)_{ten}$, XIV
11. Multiplication
12. 1
13. $(3 \cdot 7) + (4 \cdot 7) = 7 \cdot (3 + 4)$
14. Divide 2415 by 7 (or by 345)
15. a is a multiple of b, and a is either greater than b or equal to b.

Chapter 4

NON-METRIC GEOMETRY

The principal objectives of this chapter are threefold:

(1) To introduce pupils to geometric ideas and ways of thought,

(2) To give pupils some familiarity with the terminology and notation of "sets" and geometry, and

(3) To encourage precision of language and thought.

There is an attempt to guide the student to the discovery of unifying concepts as a basis for learning some of the more specific details. This chapter forms a background for Chapters 7, 9, and 10, which deal with metric or distance properties. It attempts to focus attention on ideas which are fundamental but which (while sometimes vaguely taken for granted) are often poorly understood by students.

Traditionally, these ideas have been taught when they were needed for a particular geometric discussion. But, all too often the teacher has assumed that these properties are obvious or clear without mentioning them. Also, there should be some advantage in considering together this group of closely related analagous properties and observing relations among them. The higher level study of some aspects of non-metric geometry has become a separate mathematical discipline known as projective geometry.

Reading the Text. This chapter has been written with the intent that it be carefully read by students. We suggest that not only should students be assigned to read the material, but that they also be encouraged to study it. Reading mathematics is not like reading a novel. Students may find it necessary to "get in and dig" for ideas. After Section 4-2, students should be advised to read with a paper and pencil at hand so that they may draw diagrams to assist their understanding.

Precision of Language. Pupils should be encouraged to express themselves accurately. Some pupils will be able to do much better than others.

Spatial Perception. It has been our effort here to help boys and girls develop spatial understanding. We do this in part by representations in space. We hope that suggestions given to the students and the notes here may be helpful in selecting other representations appropriate for your class. This course is not a 12th grade course. It is intended to provide background which is sometimes assumed in later courses.

Testing. In testing, try to test for grasp of ideas, not for mere recall. Students should be encouraged to express ideas in their own words. Because abilities in geometric perception and understanding differ from abilities in arithmetic, you may observe some redistribution of high and low grades among your students.

Time Schedule. The following time schedule is suggested:

1st week	Sections 1 through 4
2nd week	Sections 5 through 7 or 8
3rd week	Remainder of chapter including test evaluation.

High ability classes may do the chapter in 11 days including the test.

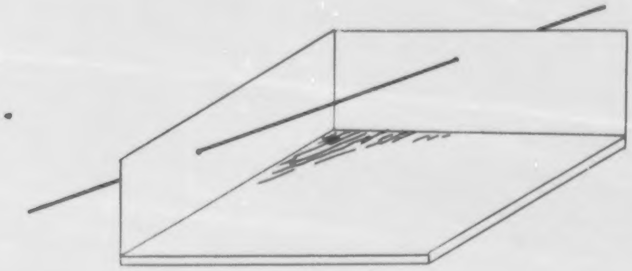
Each section was conceived as a one-day section. It seems possible that Sections 3 and 5 may take 2 days each and that an extra day or so might be spread over Sections 8, 9, 10. We believe all sections are important. We have tried to order the sections so that the ideas do not become more difficult. In fact, the second half of the chapter which deals less with spatial perception may well be easier for some students. Some students will particularly enjoy Sections 8 and 10. If a class is under great time pressure, Section 9 might be omitted. If so, it should be assigned to all the better students.

Materials.

Insights into ideas developed in this chapter will be greatly enhanced by use of instructional devices. Encourage students to make simple models as a means of helping in basic understandings. Emphasize ideas, not evaluation of models. As in using any instructional material of this kind, seek understanding of ideas without over-dependence upon representations.

Suggested Materials:

String--to represent lines in space.
Paper--to represent planes, and folded to represent lines and intersections of planes.
Tape or tacks for attaching string to walls, floor, other points in the room.
Model (as illustrated)



Suggest making the model as shown above by using a cardboard carton (or, it can be made with heavy paper, oak tag, screen wire). Cut away two sides so that only 2 adjacent sides and bottom of box remain. String, wire, etc., may be used to extend through and beyond "sides," "floor," etc.
Oak tag for making models to be used by both teacher and students.
Coat-hanger wire, knitting needles, pickup sticks.
Scissors, colored chalk.
Light-weight paper (for tracing in exercises).
Yardstick or meter stick with several lengths of string tied to it at different intervals. By fastening stick to wall, lines may be represented by holding the string taut. By gathering together the free ends at one point the plane containing the point and the yardstick may be shown.

Optional:

Long pointer for seeing lines.
Tinker-Toys for building models.
Toothpicks for student models.
Saran wrap, cellophane, and wire frame for representing planes.

4-1. Points, Lines, and Space.

Page 105.

1. Understandings:
    (a) A point has no size.
    (b) A line is a certain set of points.
    (c) A line is unlimited in extent.
    (d) Through two points there is one and only one line.
    (e) Space is a set of points.

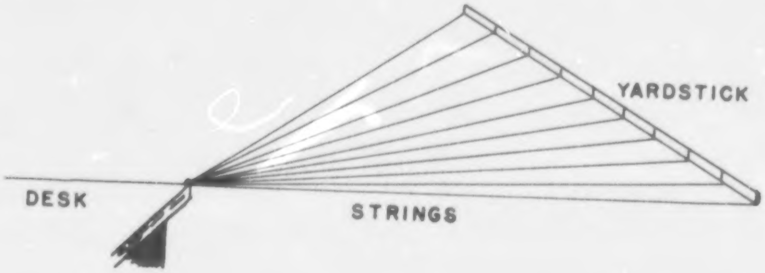
2. Teaching Suggestions:

Just as we use representations to develop the concept of the "counting numbers" (2 cars, 2 people, 2 hands, 2 balls, 2 chairs, etc., to develop the concept of twoness) similarly we must select representations for developing the concepts of point, line, plane, and space.

Point. Identify things which suggest the idea of a point keeping in mind that one suggestion by itself is not adequate for developing the idea of a point. One needs to use many illustrations in different situations. Suggestions: tip of pencil, needle, pointer; collection of boxes, putting one inside another and always being able to place one more inside of the last and the point indicated as being in all the boxes; pupil of the eye in intense brightness; progressive closure of shutter of camera; dot of light on some TV screens; particle of dust in the air.

Line. Identify two points using some of the situations as above, such as tips of two pencils, etc. Bring out the idea that given these two points there are many other points on the line that contain them. Some of these are between the two points, some are "beyond" the one, and some are "beyond" the other. Also, through two points there can be only one line. The line has no thickness and no width. It is considered to extend indefinitely. Use string held taut between two points to show representations of lines in positions that are horizontal, vertical, and slanting. Each student may represent lines by using a pencil between his fingertips. With each example talk about thinking of a line as unlimited in extent. Emphasize frequently that we use the word, "line," to mean straight line. Identify other representations of lines such as: edge of tablet (holding tablet in various positions); edge of desk; vapor trails; edges of roof of building; etc. It is important to select illustrations representing lines in space as well as the usual representations made by drawing on chalkboard and paper.

Space. Models will be most helpful here. Using "string on yardstick" and considering some point on a table, desk, or on some object which all students can see, let all the representations of lines from the yardstick pass through the point. Also, use string to show representations of other lines from other points on different walls, the floor, etc., all passing through the point. Use the model as described in drawing under "Suggested Materials." Pass lines (string, wire, thread) through "walls" and "floor" to suggest infinite number of lines and that these lines extend indefinitely. Bring out the idea that each line is a set of points, and that space is made up of all the points on all such lines.



Answers to Class Discussion Problems in 4-1.

1. Depends on what objects are in the room.
2. The body of the porcupine is like the point. The quills are like lines through this point.

Answers to Exercises 4-1. Page 108.

1. No, the ribbons are usually not straight.
2. There is exactly one line containing the two points on the monuments.
3. The line of the new string passes through the same two points as the line of the old. (Property 1)

4-2. Planes.

Page 108.

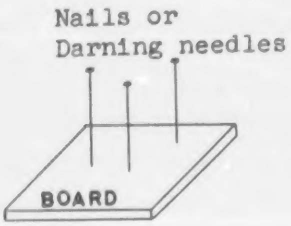
1. Understandings.
   - (a) A plane is a set of points in space.
   - (b) If a line contains two different points of a plane, it lies in the plane.
   - (c) Many different planes contain a particular pair of points.
   - (d) Three points not exactly in a straight line determine a unique plane.

2. Teaching Suggestions.

Identify surfaces in the room which suggest a plane--walls, tops of desks, windows, floor, sheet of paper, piece of cardboard, chalkboard, shadow. Make use of Saran wrap, cellophane, and a wire frame to show further a representation of a plane since this more nearly approaches the mathematician's idea of a plane. With each example bring out the idea that a plane has no boundaries, that it is flat, and extends indefinitely. It is an "ideal" of a situation just as is a line and a point. We try to give this idea by suggesting things that represent a plane. It is important to suggest representations of planes in horizontal, vertical, and slanting positions. Note that if a line contains 2 points of a plane, it lies in the plane and that many planes may be on a particular pair of points as pages of a book, revolving door, etc.

Then using three fingers or sticks of different heights in sets of 3 (not in a straight line) as suggested by the sketch at the right, see what happens when a piece of cardboard is placed on them. Add a fourth finger or a fourth stick and observe what happens.

Each student may try this experiment by using three fingers of one hand (and also three fingers using both hands) letting a plane be represented



by a book, piece of oak tag, card. Change position of fingers and thumb by bending the wrist (changing sticks in model). Ask class to make a statement about three points not in a straight line (Property 3).

Demonstrate with wires or string the ideas in the last paragraph before asking students to read it or suggest that one or two students be responsible for demonstrating the idea to other members of the class.

The Class Discussion Problems may well be developed as a class activity.

A note. What is a basic motivation for the study of geometry? In our daily living we are forced to deal with many flat surfaces and with things like flat surfaces. It would be foolish not to note similarities of these objects. So, we try to note them. In so doing we try to abstract the notion of flat surface. We try to find properties that all flat surfaces have. Thus, we are led to an abstraction of the flat surface--the geometric plane. We study two aspects of this--(1) What a plane is like, considered by itself (plane geometry), and (2) how various planes (flat surfaces) can be related in space (one aspect of spatial geometry).

Just how do we study the geometric plane? We study it by thinking of what the plane is supposed to represent, namely, a flat surface. However, in trying to understand a plane (or planes), we find it difficult to think just abstractly. Thus, we think of representations of the plane--wall, chalkboard, paper, etc., and we think of these as representations of the abstract idea. The abstract idea enables us to identify characteristics which all flat surfaces have in common.

Answers to Questions in Text, 4-2. Page 110.

Yes, this is what Property 1 says. Yes, the lines must be on the plywood. Hold palm flat and move arm and hand in plane desired. Yes, new position; and no, only one way. Three points are in one plane; the fourth may not be in it. Property 2.

Answers to Class Discussion Problems. Page 112.

1. Depends on the particular classroom.
2. There are many different sloping positions, illustrating Property 2.
   Only one, illustrating Property 3.

Answers to Exercises 4-2.

1. All in the same line.
2. Not all in the same line.
3. All legs must rest on the same plane or the device may rock. Three points not all on the same line are in exactly one plane.
4. (a) Many (unlimited quantity) (b) One
5. (a) Many (b) Many (c) Exactly one, unless the points are all on the same line.
6. Wing surface, flat surface used on water. Consult dictionary.

*7. (a) Six, as long as no three are on the same line
    (b) Six

8. Explanation in Section 4-5.

4-3. Names and Symbols.

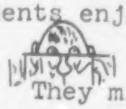
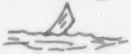
Page 113.

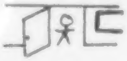
1. Understandings.
   - (a) Students learn to recognize how planes, lines on planes, lines through planes, etc. are represented by drawings.
   - (b) Students learn to name particular points, lines, and planes using letters, etc.
   - (c) Students learn how to interpret and understand perspective drawings.
   - (d) Students learn to develop an awareness of planes and lines suggested by familiar objects.

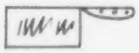
2. Teaching Suggestions.

Bring out the idea that we make agreements as to how to represent certain ideas, i.e., "." for a point; " " or "⤢" for a line; and the use of letters for naming lines and points. Note that we usually name points by capital letter, lines by lower case letters or pairs of capital letters with bar and arrows above, as, $\overleftrightarrow{AB}$, and that a plane is named by three capital letters or a single capital letter. Also, we sometimes talk about two or more lines, planes, etc., by using subscripts, such as, ℓ_1, ℓ_2, and ℓ_3.

Students should be encouraged to make such drawings on the chalkboard as well as on paper.

Students enjoy a guessing game about abstract figures such as these:  "man looking over fence" or  "boat sinking." They might enjoy a similar game with planar abstractions such as:

 "open door" or "pup tent" or

 "butcher knife."

Answers to Exercises 4-3. Page 119.

1. A is to the left; B to the right
   C is nearest the top; D nearest the bottom
2. A table
3. It has turned upside down
4. (a) Cot
   (b) Ping pong table
   (c) Football field
   (d) Carpet
   (e) High jump
   (f) Coffee table
   (g) Line of laundry
   (h) Open door
   (i) Chair
   (j) Shelf
   (k) Ladder
6. Yes (or no). We assume point V is on line $\overleftrightarrow{PQ}$ and "behind" $\overleftrightarrow{PQ}$.
   No
   No
   One, P.
7. 1, b
   2, d
   3, c
   4, d
   5, a
   6, a

8. (a) Yes
   (b) Yes
   (c) Yes, ℓ_1 is the only line through P and Q
   (d) ℓ_2 is the intersection M_1 and M_2
   (e) No
   (f) No
   (g) No

4-4. Intersection of Sets.

Page 122.

1. Understandings.

   (a) A set contains elements which are collected according some common property or explicit enumeration.

   (b) The common elements in two or more sets make up the elements of the intersection of two or more sets.

2. Teaching Suggestions.

Review the idea of sets by asking students to describe certain sets, as set of names of members of the class, the set of members of the class, set of all students in class whose last name begins with "B", set of even numbers, set of counting numbers between 12 and 70 having a factor 7 (i.e. 14, 21, 28, ...).

Explain that any 2 sets determine a set which is called their intersection, that is, the set of elements (if any) which are in both sets. Have students give the intersection for the set of odd numbers between 1 and 30 and the set of counting numbers having the factor 3 between 1 and 30. Note the three sets--the two given sets and the intersection of the two sets. Use other illustrations such as set of boys in the class and the set of students with brown eyes. Also find intersection of three sets--students in class having blonde hair, students in class having birthdays in November, and students in class riding the school bus. In selecting sets, include some in geometry (i.e. the intersection of two lines in the same plane, etc.).

Note that the empty set is the intersection of two sets with no elements in common. Call pupils' attention to the meaning of "common" as it is used here.

After developing the idea of intersection go back to examples and describe how the idea can be expressed in symbols. It is a code we can use and like many codes it simplifies the expression. For example,

Set A = {3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29}
Set B = {3, 6, 9, 12, 15, 18, 21, 24, 27}
$A \cap B$ = {3, 9, 15, 21, 27}

You may wish to refer to the supplementary unit on "sets." It will provide further symbolism for better students and/or those who especially enjoy using symbols.

Answers to Exercises 4-4. Page 123.

1. (a) {18, 19, 20, 21, 22}
   (b) As in your state
   (c) The empty set
2. (a) 1, 3, 5 (a possible answer)
   (b) 5, 10, 15 (a possible answer)
   (c) P, R, T (a possible answer)
3. (a) 9, 10, 11, 12
   (b) As in your class
   (c) The point P
   (d) The line
4. $S \cap T$ = {10, 15, 23}
5. (a) An edge
   (b) The empty set
6. (a) S = {Maine, N. H., Vt., Conn., Mass., R. I.}
   T = {New Hampshire, New York, New Jersey, N. M.}
   V = {Texas, New Mexico, Arizona, California}
   (b) New Hampshire
   (c) The empty set
   (d) New Mexico
7. (a) Yes
   (b) Yes. The addition of multiples of 5 gives multiples of 5.
8. Each part is almost obvious from the notions of set and intersection.

[page 123]

(a) This is by definition since the empty set is a set.

(b) The set of those elements in X which are also in Y is the set of those elements in Y which are also in X.

(c) Each "side" of the equality means the same set: namely, the set of all elements which are in each of the three sets X, Y, and Z.

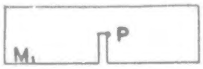
4-5. Intersections of Lines and Planes.

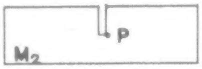
Page 125.

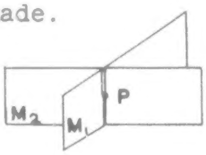
1. Understandings.
   - (a) Two lines may:
     - (1) be in the same plane and intersect;
     - (2) be in the same plane and not intersect (intersect in the empty set);
     - (3) not be in the same plane and not intersect (intersect in the empty set).
   - (b) A line and a plane may:
     - (1) not intersect (intersect in the empty set);
     - (2) intersect in one point;
     - (3) intersect in a line.
   - (c) Two different planes may
     - (1) intersect and their intersection will be a line;
     - (2) not intersect (have an empty intersection).

2. Teaching Suggestions.

Use models in order to explore the possible situations for two lines intersecting and not intersecting. (Let each student have materials, too.) Also, use a pencil or some other object to represent a line, and a card to represent a plane. Use two pieces of cardboard each cut to center with the two fitted together to represent the idea of two planes and their intersection, and, from these, state some generalizations that may be made.







Also, identify situations in the room which are representations of different cases of intersections of lines and planes. Some may wish to express the ideas in symbols. For example, name the line of one of the front vertical edges ℓ, and line along the top front of desk, k. Then $\ell \cap k$ is a point.

Subscripts also may be used so that we talk about lines ℓ_1 and ℓ_2. The use of a few subscripts should be encouraged.

Answers to Questions in Text 4-5. Page 125.

Indefinite question - no specific answer needed. Yes it is true of edges of classroom. Yes; yes; yes; no. Yes; figure from Section 4-3 could be used.

Page 126. Case 1. If they did, ℓ and k would be the same line.

Page 127.

A Line and a Plane. Yes; Figure (b) line and plane with one point intersection;

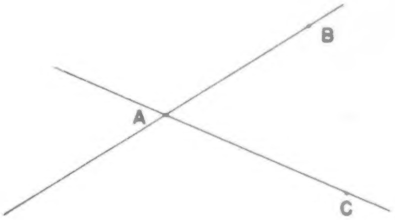
Figure (a) line lying in plane.

Two Planes. No; no, cannot hold sheets to do either.

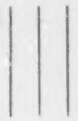
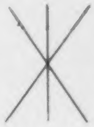
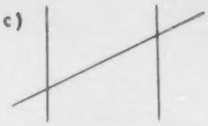
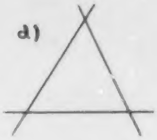
Page 128. No other case; either the planes intersect or they do not.

Answers to Exercises 4-5. Page 129.

1. (a) *Intersection of line and plane, empty set.* Line and plane are parallel. No point.
   (b) *Intersection of line and plane, one point* line pierces plane.
   (c) *Intersection of line and plane more than one point.* Line lies in plane.
2. Intersection of floor and front wall, intersection of ceiling and side wall, etc.
3.



$\overleftrightarrow{AB} \cap \overleftrightarrow{AC}$ = Point A

4. Yes; yes; no (no point in all planes); the empty set; no.
5. Yes; yes; one point
6. Yes; yes; a line (the fold)
7. (4) The empty set
   (5) One point
   (6) The line
8. a) 
   b) 
   c) 
   d) 

0 points 1 point 2 points 3 points

Using Only Lettered Points

9. (a) HGD and ABC (one other possibility)
   (b) HGB and GBC (many other possibilities)
   (c) HGB, BGD, FGD (many other possibilities)
   (d) HGD, FGD, BGD (some other possibilities)
   (e) ABC, FE (many other possibilities)
   (f) FE, GD (many other possibilities)
   (g) ED, AB (many other possibilities)
   (h) ED, GD, CD (many other possibilities)
   (i) FGB, FGD, HGD, BGD

4-6. Segments.

Page 131.

1. Understandings.
   (a) If A, B, and C are three points on a line, our intuition tells us which point is between the other two.
   (b) A line segment is determined by any two points and is on the line containing those points.
   (c) The two points which determine a segment are called endpoints of the segment.
   (d) A line segment is a set of points which consists of its endpoints and all points between them.
   (e) The union of two sets consists of all the elements of the two sets.

2. Teaching Suggestions.

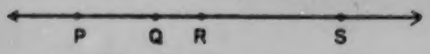
Bring out the idea that when we draw a sketch or a picture of a line, we draw a picture of one part of the line, and that this is, properly, a line segment. However, we often represent a line by a part of a line (since we cannot do anything else). One should be careful to say that the sketch represents a line or line segment as is appropriate. Draw a representation of a line on the chalkboard and name two points of the line, A and B. Note that $\overline{AB}$ means points A and B and all points between them. Name other points on the line and various line segments.

Review the idea of intersection of two sets. Then develop the idea of union of two sets using line segments much as is suggested by student text. Extend this idea by selecting other illustrations, as, set of girls in class and set of boys in class and that the union of the two sets is the set obtained by combining the two sets. Each member of the union is then either a boy or a girl in the class. Have students suggest other sets, noting what can be said about the elements of the union of the two or more sets. Take illustrations from geometry too, such as, the union of two lines, line and plane, etc. Again note that if an element is a member of both sets, it appears only once in their union

Answers to Exercises 4-6. Page 133.

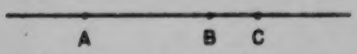
In text $\overline{BC}$

1.

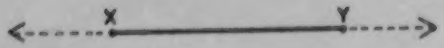


(a) $\overline{PR}$, $\overline{QS}$ or $\overline{PR}$, $\overline{QR}$ etc.
(b) $\overline{PQ}$, $\overline{QR}$ etc.
(c) $\overline{PQ}$, $\overline{RS}$
(d) $\overline{PQ}$, $\overline{RS}$

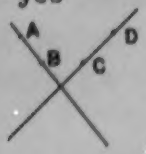
2.

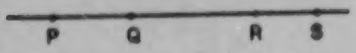


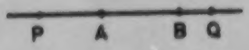
(a) The point B
(b) $\overline{BC}$
(c) $\overline{AC}$
(d) $\overline{AC}$

3. No;


yes

4.


5.


6. No.


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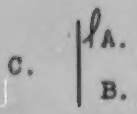


Yes; no.

8. We would have more than one "straight line" through two points. See question 6.

9. (a) This follows from the idea of set.
   (b) The set of elements which are either in X or in Y is the same as the set of elements which are either in Y or in X.
   (c) Each side of the equality means the same set; namely, the set of all elements which are in at least one of the same three sets X, Y, or Z.

10. The set of elements which are either in set X or in set X is naturally just the set X.

4-7. Separations.

Page 134.

1. Understandings.
   (a) A plane separates space into two half-spaces.
   (b) A line separates a plane into two half-planes.
   (c) A point separates a line into two half-lines.
   (d) A ray is the union of a half-line and the point which determines the half-line.

2. Teaching Suggestions.
   Use cardboard models to develop understanding of these ideas.

Note how one side of the model pictured in the materials section of this Commentary separates space and that we call the two sides half-spaces. Ask how we could separate a plane into two half-planes and a line into two half-lines before the students have read these paragraphs.

This section gives an unusually good opportunity to emphasize relations among point, line, plane, and space. You can expect seventh grade students particularly to enjoy this section. It gives a certain structure to geometry.

Draw a number of lines on the chalkboard. Mark points on them and talk about half-lines, rays, and endpoints. Talk about the intersection of 2 rays, two half-lines, and ray and half-line. If students are worried about whether a half-line has an endpoint, see explanation in student text for saying "an angle of a triangle."

Also, identify representations of half-spaces, produced by room dividers, walls in building; of half-planes, by lines on paper, lines on wall, etc.; and of half-line by naming a particular point along the edge of a ruler.

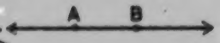
Answers to Questions in Text 4-7. Page 135.

Beginning below the first figure of the section:

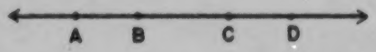
Yes; no; not the empty set. A half-line is one of the two sets of points into which a line is separated by a point. Yes, a point; yes (a set consisting of one point). No; yes; yes for figure. See statement (3) immediately below in text.

Answers to Exercises 4-7. Page 137.

1. Yes; no; yes; no; yes.
2. $\overline{AB}$ (or $\overline{BA}$); a half line to the left of A.



3.



(a) $\overrightarrow{BA}$, $\overrightarrow{BC}$
(b) $\overrightarrow{BA}$, $\overrightarrow{CD}$
(c) $\overrightarrow{BC}$, $\overrightarrow{CD}$
(d) $\overrightarrow{BA}$, $\overrightarrow{BC}$
(e) $\overrightarrow{BA}$, $\overrightarrow{CD}$

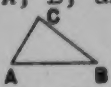
4. No, no
5. Yes, yes

6. Yes
*7. Not if they are half-planes from the same line, but yes if you use parallel lines and overlapping half-planes.
*8. No; yes

4-8. Angles and Triangles.

Page 138.

1. Understandings.
   (a) An angle is a set of points consisting of 2 rays not both on the same straight line and having an endpoint in common.
   (b) An angle separates the plane containing it.
   (c) A triangle is the union of three sets, $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ where A, B, and C are any points not on the same line.



   (d) A triangle determines its angles but does not contain its angles.

2. Teaching Suggestions.

Illustrate the idea of angle as two rays with the same endpoint. Use colored chalk to show interior and exterior. Also note how we name an angle. (This is not the only way to consider an angle. This definition, however, is identical with that which is very likely to be used in the 10th grade. Later, the measure of an angle and angle as a rotation will be considered.)

In developing the idea of triangle, put three points on board and note them as endpoints of 3 line segments, $\overline{AB}$, $\overline{BC}$, $\overline{AC}$. Note the set of points in each segment and that a triangle is the union of these three sets. Use colored chalk to show interior and exterior. Emphasize the set of points of the interior, the exterior, and that of the triangle.

Again students may be interested in drawing angles, triangles, shading, etc. (This is not perspective drawing.) This is good material to develop imagination and to encourage drawing, etc. How many triangles can be drawn with just four lines? How many with four lines two of which are parallel? Etc.

In discussing the angles of the triangle bring out the idea that although people often talk about angles of a triangle, it is a short way of saying that they are the angles determined by the triangle. Use such analogies as the school determines its graduates but graduates are not in school; a city "has" suburbs, but the suburbs are not part of the city, etc.

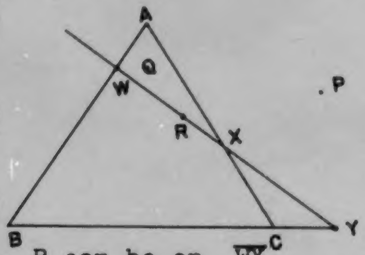
Answers to Questions in Text 4-8. Page 139.

Yes, the sides of the triangle are in the triangle; that is to say, every point of each side is in the set of points which is the triangle.

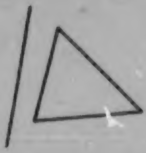
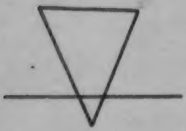
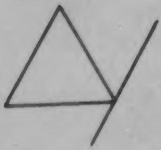
No, the angles are not in the triangle.

Answers to Exercises 4-8. Page 140.

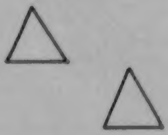
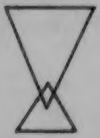
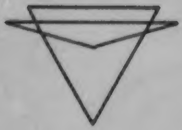
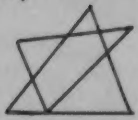
1. (a) The interior of $\angle$ ABC
   (b) The interior of $\triangle$ ABC
2. (a) Yes, they have different vertices
   (b) Yes
   (c) The lines containing the rays determine a plane.
3. (a) The point A
   (b) No
   (c) $\overline{AB}$
   (d) $\overline{AB}$
4. (a) The two points X and W
   (b) $\triangle$ AWX, $\triangle$ ABC, $\triangle$ BWY, $\triangle$ XCY
   (c) None
   (d) A, B, C, W, and Y
   (e) B, A
5. (a)



   (b)
   (c) R can be on $\overline{WX}$

6. (a)  (b)  (c) 

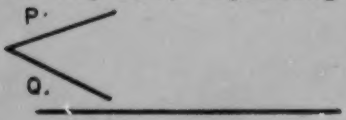
(d) Not possible

7. (a)  (b)  (c)  (d) 

8. (a) The points A and C
   (b) $\overline{AB}$
   (c) The points A and B
   (d) The point B
   (e) $\angle ACB$
   (f) $\overline{BC}$
   (g) $\overline{BC}$
   (h) The union of $\overline{AB}$ and $\overline{BC}$

*9. No; yes.

*10. (a) Yes
   (b) It may or may not, depending on choice of P and Q.
   (c) Yes
   (d) No



4-9. One-to-one Correspondence.

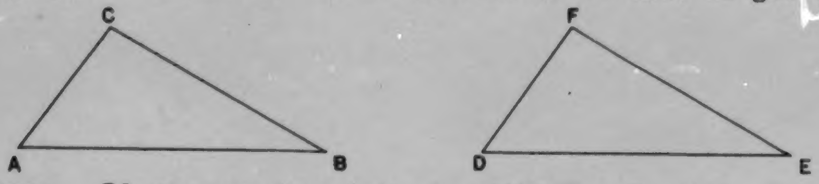
Page 142.

1. Understandings.
   (a) The idea of one-to-one correspondence is fundamental in counting. (Chapter 3)
   (b) One-to-one correspondences in geometry can be established:
      (1) Between a certain set of lines and a certain set of points
      (2) Between the set of points of one segment and the set of points of another segment
      (3) Between certain other geometric sets in pairs.

2. Teaching Suggestions.

Review the idea of one-to-one correspondence and the necessary condition that for each element in set A there corresponds an element in set B. For example, if there are 5 chairs and 5 people, for each chair there is a person and for each person there is a chair.

This idea, while elementary, is sometimes hard to grasp. One-to-one correspondences between finite sets (sets having a specific number of elements as in the illustration above) are easy to observe if they exist. If sets A and B are finite, then if A and B have the same number of elements, there exists a one-to-one correspondence between them. Such follows simply by counting. But we are sometimes interested in a particular one of the possible one-to-one correspondences. For the two congruent triangles below we are interested in matching A with D, B with E, and C with F. It is on such basis that we get the congruence. If we were to match A with F, B with D, and C with E we would not be noting the congruence.



For infinite sets H and K we may be interested in two aspects:

(1) Is there any one-to-one correspondence between H and K?

(2) Is there a "nice" or "natural" one-to-one correspondence?

In the examples in Section 4-9, we not only show that there is some one-to-one correspondence but that there is a "natural" or "nice" one. There also would be a great many that are not "natural" or "nice."

To establish a one-to-one correspondence we need (1) a complete matching scheme, and (2) in this particular device it must be true that for any element of either set there corresponds a unique element of the other set. It is implied by what we say that if a corresponds to b, then b corresponds to a.

[page 142]

In effect, to establish a one-to-one correspondence we must have a way of "tying" each element of either set to a particular element of the other. And the "string" we use for tying a to b also ties b to a.

In the text we describe a one-to-one correspondence between a certain set of lines and a certain set of points.

Problems 2 and 3 (Exercise 4-9-b) are developmental and because of this are suggested for possible classwork.

Answers to Questions in Text 4-9. Page 142.

Yes (if all ticket-holders show up). Same number.

Yes. No.

One more person than chairs. Yes, at the end of the game.

No. $$\begin{array}{ccccccc} a & b & c & d & e & f & g \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\ I & II & III & IV & V & & \end{array}$$

Answers to Exercises 4-9a. Page 143.

1. Depends on the number of pupils and the number of desks.
2. No. There are more states than there are cities of over 1,000,000 in population.
3. Yes. For each person there is a nose and for each nose there corresponds a person; hence, counting noses gives the total number of persons involved.
4. $$\begin{array}{ccccc} 1 & 3 & 5 & 7 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & 2 & 4 & 6 & \ldots \end{array}$$ each odd is matched to the next lower even.
5. Yes. Match elements of R with elements of T by means of elements in S.
6. Each whole number is matched with the whole number which is two times as large. Thus:
$$\begin{array}{cccccc} 0 & 2 & 4 & 6 & 8 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & 1 & 2 & 3 & 4 & \ldots \end{array}$$

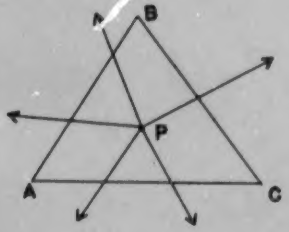
Answers to Class Exercises. Page 144.

4. (a) No.
   (b) Yes. Property 1. (Section 4-1)

5. (a) Yes.
   (b) 3.
   (c) Yes.
   (d) Yes.
   (e) Yes.
6. Yes.
7. (a) Yes.
   (b) One.
8. line; point; point; line.

Answers to Exercises 4-9b. Page 145.

1. Each vertex to the side opposite. Yes.
2.



   Yes, you should observe it.
   Yes; yes.
3. Yes; yes; yes; yes; yes.

*4. There are rays in space similar to the set of rays described in Problem 2.

*5. The set of intersections of the elements of K with a plane form the set H.

*6. (a) Yes
   (b) Yes
   (c) Yes, see Problem 5 (Exercises 4-9a).

4-10. Simple Closed Curves.

Page 147.

1. Understandings.
   (a) Broken-line figures such as those we see in statistical graphs, triangles, rectangles, as well as circles, and

figure eights are curves.

(b) A simple closed curve in the plane separates the plane into 2 sets--the points of the interior of the curve and the points of the exterior of the curve. The curve itself is contained in neither set.

(c) The curve is called the boundary of the interior (or the exterior).

(d) If a point A is in the interior of a curve and a point B is in the exterior of the curve, then the intersection of $\overline{AB}$ and the curve contains at least one element.

2. Teaching Suggestions.

Draw some curves on the chalkboard, bringing out the idea that we call them "curves" and that a segment is just one kind of curve. We use the word "curve" in a special way in mathematics.

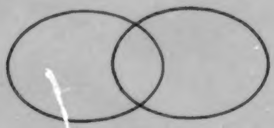
Note that a simple closed curve separates a plane into two sets and that the curve itself is the boundary of the two sets. Also, that any quadrilateral, parallelogram or rectangle is a simple closed curve. Identify some of the many curves which are suggested in the room, such as boundary of chalkboard, total boundary of floor surface, etc.

Students enjoy drawing elaborate curves which may still be classified as simple closed curves. Encourage their drawing a few simple closed curves for a bulletin board exhibit.

Answers to Exercises 4-10. Page 149.

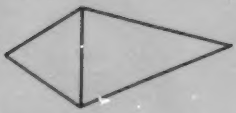
1.

Six

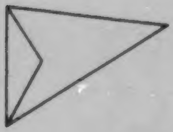


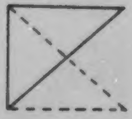
2. The intersection of the exterior of J_1 and the interior of J_2.

3.



OR



[

This is ruled out because of the extra point of intersection.]

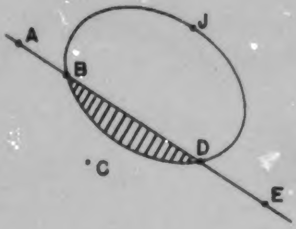
Yes, 3.

4. No, it contains two.

5. Simple closed curve, line, angle

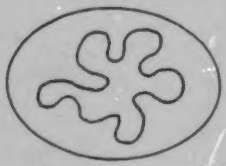
6. (a) B and D

(b)

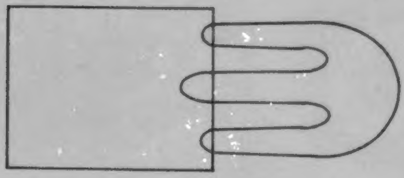


(c) $\overrightarrow{BA}$ and $\overrightarrow{DE}$

*7.



*8.

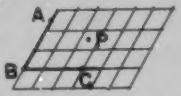


9. Because the curve would have to cross an even number of times. If the curve crosses three times, the pencil is then not on the same side of the line as where it started.

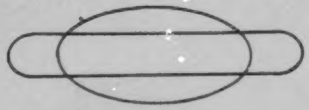
Sample Questions on Chapter 4

The sets of questions presented at the end of the chapter in the Teachers Commentary are NOT intended as a Chapter test. Teachers should construct chapter test carefully by combining selected items from the set of questions included in the guide and questions of their own writing. Great care should be used to avoid making the test too long.

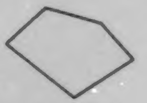
1. If two different planes each contain the same three points what can you say about the three points? (The points are on a line.)
2. Draw $\angle ABC$. Label P a point in the interior. Shade the P-side of line $\overleftrightarrow{AB}$.



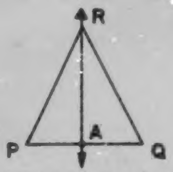
3. Draw two simple closed curves whose intersection is a set of exactly four points.



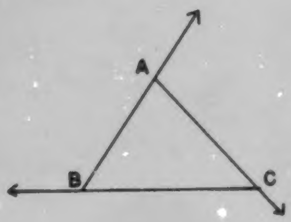
4. Draw a simple closed curve which is the union of five segments, no two on the same line.



5. Draw $\triangle PQR$. Label a point A between P and Q. Draw the line $\overleftrightarrow{AR}$. List all triangles represented in your figure. ($\triangle PQR$, $\triangle PAR$, $\triangle RAQ$.)



6. Each point of a line separates the line into two half-lines. Make two similar statements, one about a line and a plane and the other about a plane and space. (A line separates a plane into two half-planes. A plane separates space into two half-spaces.)
7. Suppose the intersection of two lines is the empty set. If they are in the same plane they are (parallel). If they are not in the same plane they are (skew).
8. Consider the figure at the right.
   (a) List 3 rays represented.
   ($\overrightarrow{BA}$, $\overrightarrow{AC}$, $\overrightarrow{CB}$)
   (b) What is $\overrightarrow{AC} \cap \triangle ABC$? ($\overline{AC}$)
   (c) What is $\overline{AB} \cap \overrightarrow{AC}$?
   (Point A)



9. Draw a horizontal line. Label four points on it P, Q, R, and S in that order from left to right.

(a) Name two segments whose intersection is one point. ($\overline{PQ}$ and $\overline{QS}$)

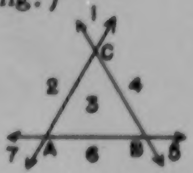
(b) Name two rays whose union is the line. ($\overrightarrow{PQ}$ and $\overrightarrow{QP}$)

(c) Name two segments whose intersection is a segment. ($\overline{PR}$ and $\overline{QS}$)

(d) Name two segments whose union is not a segment. ($\overline{PQ}$ and $\overline{RS}$)

10. Using the planes of the walls and ceiling of your room, list three planes whose intersection is exactly one point. (Front wall, side wall, and ceiling.)

11. Label A, B, and C three points not all on the same line. Draw $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$.



(a) Into how many pieces (regions) does the union of the segments $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ separate the plane? (2)

(b) Into how many pieces (regions) does the union of lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$ sepraate the plane? (7)

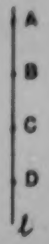
12. Draw a vertical line l on your paper. Label points A, B, C, and D in that order from top to bottom.

(a) What is $\overline{AC} \cap \overline{BD}$? ($\overline{BC}$)

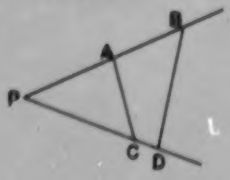
(b) What is $\overrightarrow{BA} \cap \overrightarrow{CD}$? (The empty set)

(c) What is the union of $\overline{AB}$ and $\overrightarrow{BC}$? ($\overrightarrow{AB}$)

(d) What is the union of $\overline{AD}$ and $\overline{BC}$? ($\overline{AD}$)



13. In the figure, a set of rays from P may be used to establish a (one-to-one correspondence) between $\overline{AC}$ and $\overline{BD}$.



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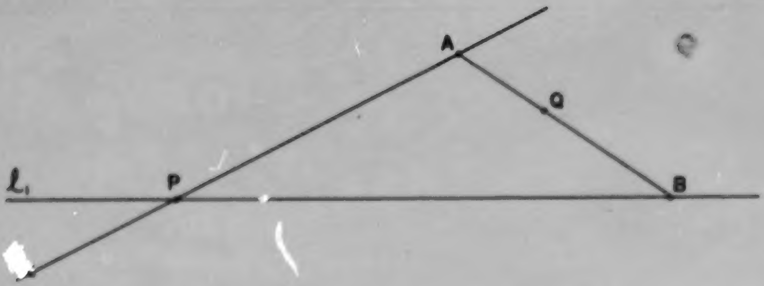
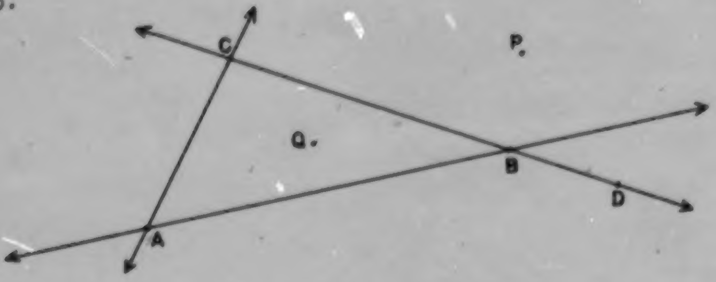


To the left of an item in the left-hand column place the letter of a corresponding item from the right-hand column:

(g) 1. the union of $\overrightarrow{PA}$ and $\overrightarrow{PB}$
(f) 2. $\ell_1 \cap \ell_2$
(c) 3. segment $\overline{AQ}$
(d) 4. point in interior of $\angle APB$
(e) 5. $\overleftrightarrow{PB}$
(a) 6. ray on ℓ_2
(b) 7. the union of $\overline{AQ}$ and $\overline{QB}$
(i) 8. $\overline{QB} \cap \ell_1$
(h) 9. $\overline{AQ} \cap \ell_1$
(j) 10. the union of ℓ_2 and $\overline{AP}$

a. $\overrightarrow{AP}$
b. $\overline{AB}$
c. $\overline{AQ} \cap \overline{AB}$
d. Q
e. ℓ_1
f. P
g. $\angle APB$
h. empty set
i. B
j. ℓ_2

15.



In the corresponding blank to the left of each of the following statements indicate if it is true or false.

(True) 1. Point Q is on the C-side of $\overleftrightarrow{AB}$.
(False) 2. $\overleftrightarrow{AB}$ is the intersection of two half-planes.
(False) 3. The union of $\overrightarrow{CD}$ and $\overrightarrow{CA}$ $\angle ABC$.
(False) 4. Point B separates $\overrightarrow{CD}$ into two segments $\overline{CB}$ and $\overline{BD}$.

(True) 5. Point B separates $\overline{CD}$ into two half-lines.

(True) 6. Point Q is in the interior of angle ACB.

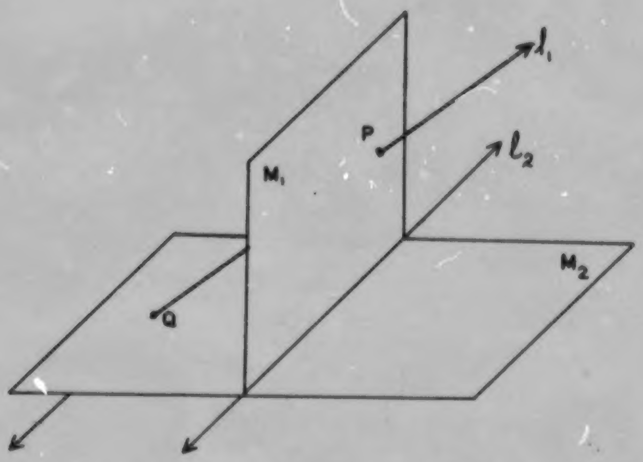
(False) 7. Point Q is in the interior of angle DBA.

(True) 8. The interior of angle ABC is the intersection of a half-plane containing C and a half-plane containing A.

(True) 9. Triangle ABC is a simple closed curve.

(True) 10. A one-to-one correspondence may be established between (a) the set of lines intersecting $\overleftrightarrow{AC}$ and containing B and (b) the set of points on $\overleftrightarrow{AC}$.

16.



Multiple Choice.

1. ℓ_1 and ℓ_2:
   - (a) intersect in the empty set
   - (b) are parallel
   - (c) intersect in one point
   - (d) are in the same plane

1 (a)

2. Segment $\overline{QP}$:
   - (a) lies in plane M_1
   - (b) is in plane M_2
   - (c) connects an element of M_1 with an element of M_2
   - (d) none of these

2 (c)

3. (a) ℓ_2 separates M_1 into two half-planes
   (b) $\ell_2 \cap M_2$ is the empty set
   (c) $\ell_2 \cap M_1$ is the empty set
   (d) ℓ_2 separates space

3 (a)

4. (a) $M_2 \cap \ell_1 = P$
   (b) $M_2 \cap M_1 = \ell_2$
   (c) M_2 does not extend endlessly as does ℓ_2
   (d) M_2 contains all the points of ℓ_1

4 (b)

5. Point P:
   (a) is in the intersection of M_1 and M_2
   (b) lies in a half-plane whose boundary is ℓ_2
   (c) is not an element of $\overline{PQ}$.
   (d) is not an element of ℓ_1

5 (b)

Chapter V

FACTORING AND PRIMES

5-1. Primes.

Page 151. It is important that in the beginning the pupils experiment with numbers and at the same time become accustomed to the equivalent terms "divisible by" and "multiple of." To emphasize this and to give the initial development of the Sieve of Eratosthenes, the pupil is first asked to cross out the multiples of 2, except for 2 itself, in the list of numbers up to 31; then the multiples of 3.

Page 152. As a class exercise, the Sieve for the numbers up to 100 is to be constructed. It will save class time for the teacher to have prepared on ditto sheets the numbers from 1 to 100 arranged in rows of ten. If the teacher wishes, the next hundred numbers might also be included. The pupils should check their copies carefully and keep them for future reference.

It should be stressed that 1 is a special number in various ways. First, every counting number is divisible by 1. Second, 1 is divisible only by itself, that is, it has only one factor. Because of these properties, the unique factorization theorem, introduced in Section 2 of this chapter, would not hold if 1 were classified as a prime number.

In constructing the Sieve the students should realize that after crossing out the multiples of 2 (except for 2) only the odd numbers remain, that is, only those numbers which are not divisible by 2. After the second step, only the numbers 1, 2, 3 and those not divisible by either 2 or 3 remain. After the third step, only the numbers 1, 2, 3, 5 and those not divisible by 2, 3, or 5 remain. After the fourth step, only the numbers 1, 2, 3, 5, 7 and those not divisible by 2, 3, 5 or 7 remain. This pattern continues.

The students are encouraged to try to see why in the fifth step, involving multiples of 11, no further numbers are crossed out in the set from 1 to 100. If no student discovers why this is so, the teacher should pass over the point since it is important that it be discovered in the student's own time; but some student may notice that any number less than 100 which is not a prime number must have a factor which is less than the square root of 100. From this it follows that, except for 1, all the numbers remaining in this Sieve after the fourth step will be prime numbers. The fourth step is the crossing out of all multiples of 7 other than 7.

Sieve of Eratosthenes

for the numbers from 1 through 100:

$\boxed{*1}$	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

for the numbers from 100 through 200:

101	~~102~~	103	~~104~~	~~105~~	~~106~~	107	~~108~~	109	~~110~~
~~111~~	~~112~~	113	~~114~~	~~115~~	116	~~117~~	~~118~~	~~119~~	~~120~~
~~121~~	~~122~~	~~123~~	~~124~~	~~125~~	~~126~~	127	~~128~~	~~129~~	~~130~~
131	~~132~~	~~133~~	~~134~~	~~135~~	~~136~~	137	~~138~~	139	~~140~~
~~141~~	~~142~~	~~143~~	~~144~~	~~145~~	~~146~~	~~147~~	~~148~~	149	~~150~~
151	~~152~~	~~153~~	~~154~~	~~155~~	~~156~~	157	~~158~~	~~159~~	~~160~~
~~161~~	~~162~~	163	~~164~~	~~165~~	~~166~~	167	~~168~~	~~169~~	~~170~~
~~171~~	~~172~~	173	~~174~~	~~175~~	176	~~177~~	~~178~~	179	~~180~~
181	~~182~~	~~183~~	~~184~~	~~185~~	~~186~~	~~187~~	~~188~~	~~189~~	~~190~~
191	~~192~~	193	~~194~~	~~195~~	~~196~~	197	~~198~~	199	~~200~~

* Remember! 1 is not a prime number.

Answers to Exercise 5-1. Page 154:

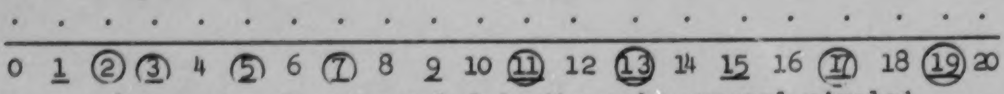
1. a. 2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71, 73,79,83,89,97
   b. 101,103,107,109,113,127
2. a. 15 b. 25 c. 31
3. 0,5,10,15,20,25,30,35,40,45,50,55,60
4. 0,7,14,21,28,35,42,49
5. 0,15,30,45,60,75,90
6.

	12	14	17	18	20	25	27
1	12	14	17	18	20	25	27
2	6	7	no	9	10	no	no
3	4	no	no	6	no	no	9
4	3	no	no	no	5	no	no
5	no	no	no	no	4	5	no
6	2	no	no	3	no	no	no
7	no	2	no	no	no	no	no

7. a. 12 = 3 × 4 or 2 × 6 b. 36 = 2 × 18 or 3 × 12 or 4 × 9 or 6 × 6 c. 31 is prime
   d. 7 is prime e. 8 = 2 × 4 f. 11 is prime
   g. 35 = 5 × 7 h. 5 is prime i. 39 = 3 × 13
   j. 42 = 2 × 21 or 3 × 14 or 6 × 7 k. 6 = 2 × 3
   l. 41 is prime m. 82 = 2 × 41 n. 95 = 5 × 19
   Of course in all these cases the two factors may be written in reverse order.
8. a. The number 24 is divisible by 1,2,3,4,6,8,12 and 24.
   b. 24 is a multiple of 1,2,3,4,6,8,12 and 24.
   c. 24 is a multiple of each of the sets of numbers in a. and b. since being a "multiple of" is the same as being "divisible by."
9. 12 = 2 × 6 = 6 × 2 = 3 × 4 = 4 × 3 = 2 × 2 × 3 = 2 × 3 × 2 = 3 × 2 × 2
10. 3 and 5, 5 and 7, 11 and 13, 17 and 19, 29 and 31, 41 and 43, 59 and 61, 71 and 73. There are eight such pairs.

11. $4 = 2 + 2$; $6 = 3 + 3$; $8 = 5 + 3$; $10 = 3 + 7 = 5 + 5$; $12 = 5 + 7$; $14 = 3 + 11 = 7 + 7$; $16 = 3 + 13 = 5 + 11$; $18 = 5 + 13 = 7 + 11$; $20 = 3 + 17 = 7 + 13$; $22 = 3 + 19 = 5 + 17 = 11 + 11$.

12. Yes; 3,5,7. This is the only set because at least one of any set of three consecutive odd numbers is divisible by 3.

13.


etc. (d) yes. The numeral 2 is the only numeral circled which is not underlined.

14. 3,5,7,11,13,17,19,23,29.

5-2. Factors.

The purposes of this section are to develop understandings of the terms "factor," "complete factorization," "composite number," and of the Unique Factorization Property. Notice that the definition of factor includes zero; thus zero has factors. Problem 4 of Exercises 5-2 serves as a basis of a discussion of zero.

Page 157. We do not consider it mandatory that pupils use exponents. They should be encouraged, however, to use them where-ever it is reasonable to do so.

As a matter of interest, the Unique Factorization Property is usually referred to as the Fundamental Theorem of Arithmetic.

Notice that in complete factorization for a prime number like 17 only one factor is written, as: $17 = 17$.

The definition given of a composite number is felt to be the easiest to understand of various possible ones. Some teachers may prefer to stress the definitions in terms of number of factors since this does have some unifying effect. If this is the case, the counting numbers would be classified as follows:

a. The number 1 which has just one factor, namely, itself.
b. The prime numbers which have just two factors; namely, 1 and the number itself
c. The composite numbers which have more than two factors.

It is important to notice that every counting number comes under one of these headings.

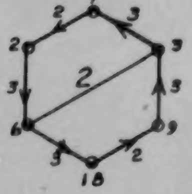
Notice that the definition is given of a complete factorization, not the complete factorization, since the order of the prime factors may vary.

In the previous edition of this section, diagrams of numbers were considered in connection with factoring. Teacher reaction was mixed, some finding it very useful and others of no assistance at all. Teacher reactions seem to indicate that some students acquire better understanding of factorization by use of these diagrams, others are confused by them. Since also it was felt that the development of this method might be easier if the teacher was not bound by a text, we are confining the description of this method to the teacher's commentary.

Basically, we can represent a product by the diagram $\overset{2}{\circ}\xrightarrow{3}\overset{6}{\circ}$ where the 3 associated with the line is the multiplier, which "takes 2 into 6." The arrow indicates the direction in which the multiplication goes.

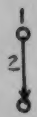
Similarly $\overset{2}{\circ}\xrightarrow{3}\overset{6}{\circ}\xrightarrow{4}\overset{24}{\circ}$ represents the product (2 × 3) × 4.

Then the three different complete factorizations of 18 could be represented by the following diagram. Notice that all the factors of 18 also appear in the diagram



In making a diagram you may suggest to the pupils that it is best first to make a list of all the factors of the number and to arrange them from the smallest to the largest. For 12 this would be 1, 2, 3, 4, 6, 12. Then start with 1 and continue to build a chain so that the second number is divisible by the preceding number, and so on. Thus, one chain would consist of 1, 2, 4, and 12; another chain would consist of 1, 3, 6, and 12; and the last chain would be 1, 2, 6, and 12. In each of these chains, taking any pair, the second is divisible by the first and there is no factor between them. We could not go from 1 to 4 since there is the factor 2 between 1 and 4. Remember that one of the rules of the game is that there may be no other factor between successive numbers.

The diagrams for the numbers from 1 through 20 are shown here:



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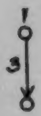

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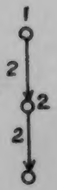

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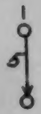

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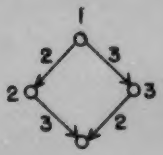

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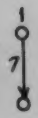

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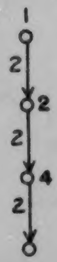

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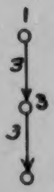

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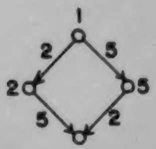

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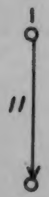

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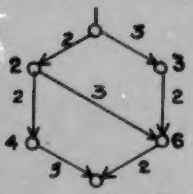
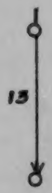

1
2
3
2
3
2
3
2
4
6
3
2

12



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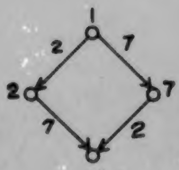

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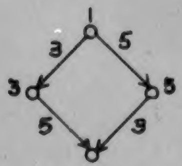

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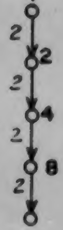
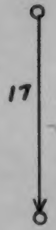
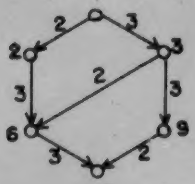


16



17


1
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2
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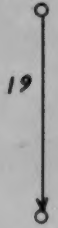
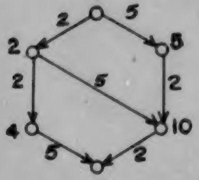
3
6
9
3
2

18



19


1
2
5
2
5
2
5
2
4
10
5
2

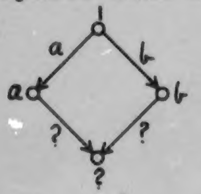
20

[pages 154-157]

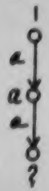
Some teachers may be interested in pursuing the investigation of these diagrams a little further. The following examples are included for this purpose. Also, a few of your very gifted pupils may gain better understandings about numbers from these examples. However, it is felt that if presented to all students, the extra time used may not justify the results.

Example I. Let a and b represent two different prime numbers. Complete each of the following sketches:

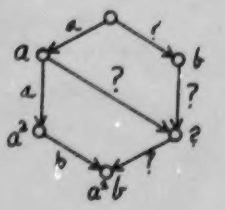
a.



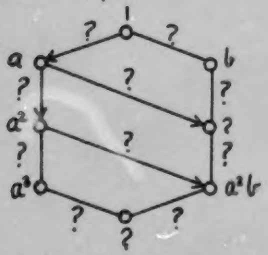
b.



d.



d.



Example II.

a. We found that 6 and 10 had patterns like the one in Problem 5a. Name 3 other numbers that we have not sketched which have the same pattern.

b. Notice that 4 and 9 have the same pattern as Problem 5b. Name 3 others which we have not sketched that have the same pattern.

c. A number like 12 or 18 has the same pattern as Problem 5c. Find 3 other numbers which have this pattern.

d. Find 3 numbers which have patterns like 5d.

e. Find patterns which have not been represented so far in the unit.

Answers for Example I.

a. ab b. a^2 c. ab d. ab; a^3b

Answers for Example II.

a. 21, 22, 26, 33, 34, and so on.

[pages 154-157]

b. 25, 49, 121 or any other square of a prime number.

c. 28, 44, 45 or any others of the form a^2b where a and b are prime.

d. 24, 40, 54, 135 or any others of the form a^3b.

e. Here are some possible forms: a^2b^2: 36, 100, 225; a^3b^2: 72, 200, 108

Answers to Exercises 5-2 Page 158:

1. a. 1, 2, 5, 10
   b. 1, 3, 5, 15
   c. 1, 3, 9
   d. 1, 2, 3, 6, 9, 18
   e. 1, 3, 9, 27
   f. 1, 2, 3, 4, 6, 8, 12, 24
   g. 1, 11
2. a. $1 \cdot 10$; $2 \cdot 5$
   b. $1 \cdot 15$; $3 \cdot 5$
   c. $1 \cdot 9$; $3 \cdot 3$
   d. $1 \cdot 100$; $2 \cdot 50$; $4 \cdot 25$; $5 \cdot 20$; $10 \cdot 10$
   e. $1 \cdot 24$; $2 \cdot 12$; $3 \cdot 8$; $4 \cdot 6$
   f. $1 \cdot 16$; $2 \cdot 8$; $4 \cdot 4$
   g. $1 \cdot 72$; $2 \cdot 36$; $3 \cdot 24$; $4 \cdot 18$; $6 \cdot 12$; $8 \cdot 9$
   h. $1 \cdot 81$; $3 \cdot 27$; $9 \cdot 9$
3. a. $2 \cdot 5$
   b. $3 \cdot 5$
   c. $3 \cdot 3$ or 3^2
   d. $2 \cdot 3 \cdot 5$
   e. $3 \cdot 3 \cdot 5$ or $3^2 \cdot 5$
   f. $2 \cdot 5 \cdot 5$ or $2 \cdot 5^2$
   g. 13
4. Zero is not a factor of six since there is not a number which, when multiplied by zero, gives a product of six, 6 is a factor of zero since the product of six and zero is zero, thus the definition is satisfied.

[page 158]

5. a. 1, 4, 10, 20 b. 1, 4, 6, 8, 9, 18, 24, 36, 72

6. a. $3 \cdot 5 \cdot 7$
   b. $2 \cdot 3 \cdot 7$
   c. $3 \cdot 5 \cdot 5$ or $3 \cdot 5^2$
   d. $3 \cdot 2 \cdot 2 \cdot 5 \cdot 5$ or $3 \cdot 2^2 \cdot 5^2$
   e. 2^6
   f. $3 \cdot 5 \cdot 23$
   g. 311 (This is a prime number)
   h. $2^3 \cdot 5^3$
   i. $7 \cdot 43$
   j. $17 \cdot 19$

7. a. Even
   b. Even
   c. Even
   d. Even
   e. Odd
   f. Even
   g. Odd
   h. Odd
   i. Odd
   j. Even

8. a. 11_{three} is four; it is even
   b. 12_{five} is seven; it is odd.
   c. 33_{five} is eighteen; it is even.
   d. 101_{two} is five; this is odd.

9. Divisibility is a property of number. It is the number which is divisible by another number. The numeral is a way of writing the number. In base ten, a numeral which represents an even number ends with an even number; in base five this is not necessarily so. This is illustrated in Exercise 8.

*10. Some of the items in the table will be:

	Factors of N	Number of Factors	Sum of Factors
18	1,2,3,6,9,18	6	39
24	1,2,3,4,6,8,12,24	8	60
28	1,2,4,7,14,28	6	56

   a. 2,3,5,7,11,13,17,19,23,29 (the prime numbers)
   b. 4,9,25 (the squares of prime numbers)
   c. Three: $1, p$ and p^2
   d. Four: $1, p, g, pg$. The sum is $1 + p + g + pg$
   e. The factors are: $1, 2, 2^2, 2^3, \ldots, 2^k$. There are $k + 1$ of them.

f. The factors are: $1, 3, 3^2, 3^3, \ldots, 3^k$. There are $k + 1$ of them.

g. If $N = p^k$, the factors are $1, p, p^2, p^3, \ldots, p^k$. There are $k + 1$ of them.

h. In this list the only numbers having $2N$ for the sum of their factors are 6 and 28. These are the two smallest perfect numbers. There is a general formula for all even perfect numbers. It is

$2^{p-1}(2$ [illegible] $1)$ where both p and $2^p - 1$ are prime numbers. The third perfect number is 496.

5-3. Divisibility.

Page 160. The purpose of this section is to facilitate the discovery of factors of numbers by inspection of the decimal system numerals which represent them. Since we need be concerned only with prime factors, we confine ourselves chiefly to 2,3 and 5. At this point in the pupil's development it was felt that the discovery of patterns would be more meaningful than the discussion dissecting the numeral according to the powers of ten. Some teachers may prefer the treatment in the supplementary unit on divisibility since it gives a little more insight into the reasons back of these patterns.

It is hoped that most of the pupils will go a little further than merely seeing that as far as the table goes, every number divisible by 3 has a sum of digits which is also divisible by 3. They should look at the transition where the tens digit increases by 1, as is suggested in Problem 3. Some may be able to go further and see that where the hundreds digit increases by 1, the tens digit decreases by 9 and the unit digit by 7, leaving a net decrease of $9 + 7 - 1 = 15$ which is a multiple of 3. Some pupils may even go further and notice that if one takes the sum of the digits, the same property holds.

It is not intended that great stress shall be laid here on tests for divisibility in other bases but it is very important that the pupils realize that the tests given here depend on the way the number is written (that is, the system of numeration used). Just a little discussion of tests for other number systems seems to be the best way to emphasize this point.

Answers to Exercises 5-3 Page 163:

1. a. 5   b. 3   c. 3   d. 2   e. 7   f. 11
2. a. $39 = 3 \times 13$ e. $180 = 2^2 \times 3^2 \times 5$ i. $576 = 2^6 \times 3^2$
   b. $60 = 2^2 \times 3 \times 5$ f. $2 \times 3 \times 43$ j. $729 = 3^6$
   c. $81 = 3^4$ g. $378 = 2 \times 3^3 \times 7$ k. $1098 = 2 \times 3^2 \times 61$
   d. $98 = 2 \times 7^2$ h. $432 = 2^4 \times 3^3$ l. $2324 = 2^2 \times 7 \times 83$
3. See discussion above.
4. For divisibility by 5 one needs only to see that when 5 is added to a number whose units digit is 0, the sum has units digit 5; if 5 is added to a number whose units digit is 5, the sum has units digit 0. This pattern repeats to show the test for all numbers.
5. The following table can be made for multiples of 9:

Multiple of 9	9	18	27	36	45	...90	99	108...
Sum of digits	9	9	9	9	9	9	18	9

6. A counting number will be divisible by 6 if and only if it is divisible by both 2 and 3. Hence the test is that it must be even and the sum of its digits must be divisible by 3.
7. If a number is divisible by 15 it must be divisible by both 3 and 5, and conversely. Hence the test is that its last digit must be one of 0 and 5 and the sum of its digits must be a multiple of 3.
8. a. Since the last digit is odd, the number is odd.
   b. The number is 400 in the decimal system and hence is even. The bright pupil might notice that the number is even since it can be written in the form $7^3 + 7^2 + 7 + 1$ which is the sum of an even number of odd numbers.
   c. Here the number in the decimal system is 259 which is odd. The bright student might notice that it may be written $6^3 + 6^2 + 6 + 1$ which is the sum of 1 and three even numbers; hence is odd.

d. Here the number in the decimal system is 40 which is even; or it may be shown even by the same kind of argument as in part b.

9. If a number is written in the system to the base seven, its last digit is zero if and only if it is divisible by seven, but it need not be divisible by ten. To test divisibility by 3 write the first few multiples of three in the system to the base seven as follows:

Number to the base seven	3	6	12	15	21	24	30	33	36	42
Sum of the digits	3	6	3	6	3	6	3	6	12*	6

*Notice that 12 is also written in the system to the base seven. Here when the first digit increases by 1, the second digit decreases by 4 giving a net decrease of 3. Hence the same test for divisibility by 3 works both in the decimal system and in the system to the base seven.

10. If a number is written in the number system to the base twelve and has zero as its last digit it must be divisible by 12 but need not be divisible by 10. It will be divisible by 3 if its last digit is one of 0,3,6,9. This may be shown in the same way that we tested for divisibility by 5 in the decimal system, since the pattern 3,6,9,0 repeats in the sequence of multiples of 3 written to the base twelve.

11. A number written to the base seven will be divisible by 6 if the sum of its digits is divisible by 6. This is apparent from the table given for exercise 9 if we notice that every other sum of digits is even.

12. A number written in the decimal system will be divisible by 4 if one of the following holds:
    a. the last digit is one of 0,4,8 and the tens digit is even.
    b. the last digit is 2 or 6 and the tens digit is odd.

    This can be seen from the pattern in which the multiples of 4 fall. Also, since any multiple of 100 is divisible by 4, we could also say that a number is divisible by 4 if the number represented by the last two digits is divisible by 4.

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5-4. Greatest Common Factor.

The following skills and understandings should be developed through this section:

1. To identify the common factors of several numbers.
2. To find the greatest common factor of several numbers in two ways--by listing the factors of each and by using a complete factorization of each number.

It is recommended that the abbreviation (G.C.F.) not be used in this section. Use the complete phrase "greatest common factor" in order to fix the meaning of the concept in the minds of the pupils.

Explain that if something is "common" to several things, then one meaning is that they are similar in some way. If two numbers have a certain number as a factor, we say that factor is common to both.

Provide drill on finding the greatest common factor by listing all factors to ensure that pupils have a thorough understanding of the meaning of the term. Some students may discover the method of finding the greatest common factor by complete factorization before it is introduced in problem 14, but if they do not discover it, do not "push" them into this prematurely. Students should be encouraged to use "short division" when the divisor is less than ten.

Discussion of Exercises 5-4. Page 166.

14-16. Method of finding the Greatest Common Factor by Complete Factorization. This is a procedure for computing the greatest common factor but it is not intuitive. The method of the definition (listing all the factors, etc.) should be used in class until the concept is fixed in the minds of the pupils.

Illustration of Method: Find the greatest common factor of 24 and 60.

First, find a complete factorization of each number:

$24 = 2^3 \times 3$

$60 = 2^2 \times 3 \times 5$

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The factor 2 occurs three times in the complete factorization of 24 and twice for 60; hence, 2 can occur at most twice for 60; hence, 2 can occur at most twice in any factor common to both 24 and 60. The factor 3 occurs just once in each, and hence can occur at most once in any common factor. The number 5 is not a factor of 24, and hence cannot occur in any common factor. Hence the greatest common factor is $2^2 \times 3 = 12$. Notice that the power of each prime factor of the greatest common factor is the smaller of the powers to which it occurs in the two given numbers.

Summary of Method: (1) Find a complete factorization of each number. (2) Pick out the primes which occur in all (every one) of these factorizations. (3) For each prime picked in (2), write the smallest power of it which occurs in any of the factorizations. (4) Multiply all the powers written in (3); this product is the greatest common factor.

The pupils will find it easier to pick out the primes and exponents in steps (2) and (3) above if the original numbers are written in a column with the complete factorization of each number to the right of the number.

16. h. This exercise gives the pupils an opportunity to establish a procedure for searching for factors of numbers when they cannot guess them. It is time consuming, and may be omitted if time is not available for it.

In factoring 1173, remind the pupils of the test for divisibility by 3 and obtain $1173 = 3 \times 391$. For the number 391, we try the successive primes 2, 3, 5, 7, 11, 13, 17, to find $391 = 17 \times 23$. Review the reasons for trying only the primes--i.e. for skipping 9, 15, and all the even numbers except 2. Point out that, since $20 \cdot 20 = 400$ is larger than 391, it is only necessary to try the primes less than 20 (391 cannot be the product of two factors each of which is larger than 20). In general, if the primes are tried in order of magnitude, one should stop when the quotient is less than or equal to the divisor.

18. Illustrate the results in a , b , c below using the values $a = 6$, $b = 9$, $c = 12$.

    a. The set of counting numbers is closed under the operation $\triangle$ since, for any counting numbers a and b, $a \triangle b$ = the greatest common factor of a and b is again a counting number.

    b. The operation $\triangle$ is commutative since the greatest common factor of a and b is the same as the greatest common factor of b and a.

    c. The operation $\triangle$ is associative since each of $a \triangle (b \triangle c)$ and $(a \triangle b) \triangle c$ is the greatest common factor of the set {a, b, c}.

Answers to Exercises 5-4.

1. a. {1, 2, 3, 6}
   b. {1, 2, 4, 8}
   c. {1, 2, 3, 4, 6, 12}
   d. {1, 3, 5, 15}
   e. {1, 2, 4, 8, 16}
   f. {1, 3, 7, 21}
2. a. {1, 2}
   b. {1, 2, 4}
   c. {1, 3}
   d. {1, 2}
   e. {1, 3}
   f. {1, 2, 4}
3. a. {1, 19}
   b. {1, 2, 4, 7, 14, 28}
   c. {1, 2, 3, 4, 6, 9, 12, 18, 36}
   d. {1, 2, 4, 5, 8, 10, 20, 40}
   e. {1, 3, 5, 9, 15, 45}
   f. {1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72}
4. a. {1}
   b. {1, 2, 4}
   c. {1, 2, 4}
   d. {1, 3, 9}
   e. {1, 2, 4, 8}
   f. {1}
5. a. 4
   b. 4
   c. 4
   d. 9
   e. 8
   f. 4

6. a. 5
   b. 6
   c. 12
   d. 25
   e. 16
   f. 3
   g. 12
   h. 8
   i. 15
   j. 10

7. a. 6
   b. 29
   c. a

8. a. 1
   b. 1
   c. 1

9. a. Yes, 1
   b. Yes, $c = 3$, $a = 3$, $b = 6$ or 9 or 12, etc.
   c. No; the greatest common factor can never be greater than the smallest member of the set of numbers used.

10. a. No. {6, 10, 15}
    b. Yes. G. C. F. of 6 and 10 is 2.
    G. C. F. of 6 and 15 is 3.
    G. C. F. of 10 and 15 is 5.

11. a. A is {1, 2, 3, 6, 9, 18}
    b. B is {1, 2, 3, 6, 7, 14, 21, 42}
    c. {1, 2, 3, 6}
    d. {1, 2, 3, 6}
    e. The intersection set of the set of factors of A and the set of factors of B is the set of common factors.

12. a. C is {1, 2, 3, 5, 6, 10, 15, 30}
    b. D is {1, 3, 17, 51}
    c. $C \cap D$ is {1, 3}

13. a. E is {1, 3, 13, 39}
    b. G is {1, 2, 4, 13, 26, 52}
    c. $E \cap G$ is {1, 13}

14. a. $36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$ (See Discussion of Exercises 5-4)
    $45 = 3 \cdot 3 \cdot 5 = 3^2 \cdot 5$
    b. The greatest common factor is 3^2 or 9.

15. a. $18 = 2 \cdot 3 \cdot 3 = 2 \cdot 3^2$
    b. $90 = 2 \cdot 3 \cdot 3 \cdot 5 = 2 \cdot 3^2 \cdot 5$
    c. $GCF = 2 \cdot 3^2 = 18$

16. a. $24 = 2^3 \cdot 3$, $60 = 2^2 \cdot 3 \cdot 5$
    G.C.F. $= 2^2 \cdot 3 = 12$
    b. $36 = 2^2 \cdot 3^2$, $90 = 2 \cdot 3^2 \cdot 5$
    G.C.F. $= 2 \cdot 3^2 = 18$
    c. $72 = 2^3 \cdot 3^2$, $108 = 2^2 \cdot 3^3$
    G.C.F. $= 2^2 \cdot 3^2 = 36$
    d. $25 = 5^2$ $75 = 3 \cdot 5^2$ $125 = 5^3$
    G.C.F. $= 5^2 = 25$
    e. $24 = 2^3 \cdot 3$, $60 = 2^2 \cdot 3 \cdot 5$, $84 = 2^2 \cdot 3 \cdot 7$
    G.C.F. $= 2^2 \cdot 3 = 12$
    f. $42 = 2 \cdot 3 \cdot 7$, $105 = 3 \cdot 5 \cdot 7$, $147 = 3 \cdot 7^2$
    G.C.F $= 3 \cdot 7 = 21$
    g. $165 = 3 \cdot 5 \cdot 11$, $234 = 2 \cdot 3^2 \cdot 13$
    G.C.F. $= 3$
    h. $306 = 2 \cdot 3^2 \cdot 17$, $1173 = 3 \cdot 17 \cdot 23$
    G.C.F. $= 3 \cdot 17 = 51$ (See Discussion of Exercises 5-4.)
    i. $2040 = 2^3 \cdot 3 \cdot 5 \cdot 17$, $2184 = 2^3 \cdot 3 \cdot 7 \cdot 13$
    G.C.F. $= 2^3 \cdot 3 = 24$

17. a. 6 ($6 \times 0 = 0$)
    b. 1
    c. 1

18. a. Yes (See Discussion of Exercises 5-4)
    b. Yes
    c. Yes

5-5. Remainders in Division.

This section serves a dual purpose: to help review basic understandings about division and to separate the section dealing with greatest common factor and least common multiple.

Also, this section should help strengthen the pupils' ideas concerning divisibility. We use the idea of the Euclidean division algorithm, but we do not introduce the term "algorithm."

[pages 168-169]

It is not necessary to confuse pupils with terms not needed in understanding the basic ideas discussed.

Page 170. The division relation should be discussed following the introductory work and review. It is important that the pupil understand the relationships among a, b, q, and R in the expression $a = (b \cdot q) + R$. If care is used to relate these symbols with the terms dividend, divisor, quotient, and remainder, the pupil should not find it difficult to use the symbol form with understanding.

The pupil should understand that division always involves a remainder. We may not always write a zero remainder, but the importance of recognizing zero as a remainder should not be minimized. It should be pointed out to students that when the remainder is zero we can use the division relation to express divisibility.

Discussion of Exercises 5-5.

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1. j. It should be noted that there are several possible answers. This example may be used to help explain the answers in question 2.

7. This method is based on the division algorithm. Most students will not understand all that is involved here. It is recommended that teachers not try to achieve mastery of this technique. This is a starred exercise and may be omitted without serious loss. It is introduced to review greatest common factor and to justify the discussion of the division relation in this chapter.

Answers to Exercises 5-5. Page 172.

1.	Dividend	Divisor	Quotient	Remainder
a.			2	
b.			4	2
c.		9		
d.			7	2
e.			4	2

	Dividend	Divisor	Quotient	Remainder
f.			3	2
g.		10		
h.	66			
i.		20		
	There are several possible answers for j.			
j.		9	9	
		3	27	
		1	81	
		27	3	
		81	1	

2. a. No; see a and j.
   b. The dividend is greater than the quotient in this case.
   c. The divisor must always be greater than the remainder.
   d. Yes. $0 \div 3 = 0$ or $0 = 3 \cdot 0 + 0$.
   e. No. $3 \div 0$ is impossible because there is no number which when multiplied by 0 gives 3, with a remainder less than the divisor.
   f. Yes. $0 \div 3 = 0$
   or $3 \div 5$ may be considered as giving a quotient of 0 with remainder 3. This might be the answer to the question: "How many \$5 shirts can you buy with \$3?"
   g. Yes. $6 \div 6 = 1$ with 0 remainder
   $6 \div 2 = 3$ with 0 remainder
3. a. Yes
   b. No. It is impossible to divide by 0.
   c. Yes.
   d. Yes. A remainder of 0 is a whole number.

4.	a	b	q	R
a.		2		1
b.	98			
c.			4	2

4. con't

	a	b	q	R
d.		1	100	
		2	50	There are many possible answers as indicated here. By the commutative property the reverse order for each of these is also an acceptable answer.
		4	25	
		5	20	
		10	10	
e.			16	11
f.		25		

5. a. No

b. Yes, (16 ÷ 2 = 8, 6 = 2, q = 8)

c. Yes, (200 ÷ 75 = 2 and Remainder 50
q = 2, R = 50).

d. No. The divisor, b, may be any whole number except zero. Division by zero is impossible.)

e. Yes (counting numbers do not include 0).

f. Yes. (a may be 0, then q = 0. Or, a, may be any other number. However, q is not a counting number if a < b.)

6. a. {0, 1, 2, 3}

b. The members of the set of all remainders are the whole numbers less than eleven.

c. 25

d. K

7. a. (1) 92 ÷ 32 = 2 and Remainder 28

(2) 32 ÷ 28 = 1 and Remainder 4

(3) 28 ÷ 4 = 7 and Remainder 0

4 is the divisor that results in a zero remainder.
The G.C.F. is 4.

b. (1) 192 ÷ 81 = 2 and Remainder 30

(2) 81 ÷ 30 = 2 and Remainder 21

(3) 30 ÷ 21 = 1 and Remainder 9
(4) 21 ÷ 9 = 2 and Remainder 3
(5) 9 ÷ 3 = 3 and Remainder 0
3 is the divisor that results in a zero remainder.
Therefore the G.C.F. is 3.

c. (1) 150 ÷ 72 = 2 and Remainder 6
72 ÷ 6 = 12 and Remainder 0
The G.C.F. is 6.

d. (1) 836 ÷ 124 = 6 and Remainder 92
(2) 124 ÷ 92 = 1 and Remainder 32
(3) 92 ÷ 32 = (see answer 7(a). The G.C.F. is 4.

e. G.C.F. is 28

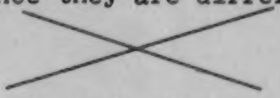
f. G.C.F. is 71

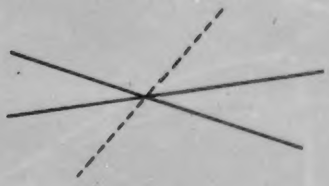
5-6. Review.

This section provides a review of the work of grades 1-6 and of chapters 1-4 of this book. It is placed here to allow the pupils extra time to fix in their minds the concept of greatest common factor before being introduced to least common multiple.

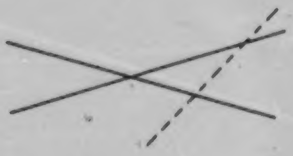
Discussion of Exercises 5-6.

9. a. Consider first two of the lines. Since they are different lines and are not parallel, they intersect in exactly one point as sketched. The third line either goes through this point of intersection or it doesn't. The two cases are illustrated below.





1 intersection



3 intersections

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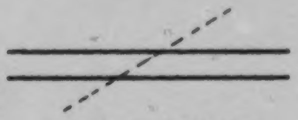
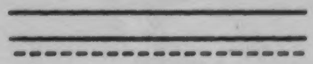
b. c. In part a , we discussed the cases where no two of the lines are parallel; these two cases are also possible in parts b and c. We have left to consider the case where at least two lines are parallel. The third line is either parallel to both of them or to neither of them. The two cases are illustrated below.


b. 2 intersections


c. 0 intersections

19. Since a, b, f, g are four different numbers and g is a factor of both a and b, we must have $g < a$, $g < b$. Also, f is a factor of both a and b (different from g) and g is the largest common factor of a and b. Thus $f < a$, $f < b$ and $f < g$. Illustrate with $a = 12$ $b = 18$, $f = 3$, $g = 6$.

20. If the numbers are allowed to be equal, all four of them may be the same so that there may be no inequalities among them. Illustrate with $a = b = f = g = 4$.

22. a. If we consider the experiment of drawing one piece of paper from the hat, there are 50 possible outcomes: the numeral drawn could represent any one of 1, 2, 3, ..., 50. Of these 50 outcomes there are five favorable (10, 20, 30, 40, 50). Thus the probability of a favorable outcome is $\frac{5}{50} = \frac{1}{10}$.

b. Of the 50 outcomes, there are 10 which are favorable (the numeral drawn represent a number divisible by 5), namely: 5, 10, 15, ..., 50. Thus the probability of a favorable outcome is $10/50 = 1/5$.

The explanation for c and d follows from above.

23. a. Of the 50 possible outcomes, there are 50 in which the numeral drawn represents a number which is divisible by 1, namely: 1, 2, 3, ..., 50. Since every numeral represents a number divisible by 1, only one draw is needed.

b. For 2, we would have to draw enough times to draw all numerals representing numbers not divisible by 2; 1, 3, 5, ..., 49. There are twenty-five such numerals. One more, the twenty-sixth, must be a number divisible by 2.

c. Of the 50 possible outcomes, there are 16 in which the numeral drawn represents a number which is divisible by 3, namely: 3, 6, 9, ..., 48. To be sure of drawing at least one of these numerals, we would have to draw enough times to draw all the other numerals and then one more; i.e. (50 - 16) + 1 = 35 draws.

24. a. Each of the sets has eleven elements (or members). One suggested correspondence is given here:

$$\begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array}$$

b. Each of the sets {51, 53, 55, ..., 79} and {18, 20, 22, ..., 46} has 15 elements. There are many different ways in which a 1-1 correspondence can be set up between the two sets. Three of the correspondences are given below.

$$\begin{array}{lcl} 51 \longleftrightarrow 18 & \qquad 51 \longleftrightarrow 46 & \qquad 51 \longleftrightarrow 20 \\ 53 \longleftrightarrow 20 & \qquad 53 \longleftrightarrow 44 & \qquad 53 \longleftrightarrow 22 \\ 55 \longleftrightarrow 22 & \qquad 55 \longleftrightarrow 42 & \qquad 55 \longleftrightarrow 24 \\ \cdot \qquad \cdot & \qquad \cdot \qquad \cdot & \qquad \cdot \qquad \cdot \\ \cdot \qquad \cdot & \qquad \cdot \qquad \cdot & \qquad \cdot \qquad \cdot \\ \cdot \qquad \cdot & \qquad \cdot \qquad \cdot & \qquad \cdot \qquad \cdot \\ 79 \longleftrightarrow 46 & \qquad 79 \longleftrightarrow 18 & \qquad 77 \longleftrightarrow 46 \\ & & \qquad 79 \longleftrightarrow 18 \end{array}$$

c. Each of the sets {3, 6, 9, ..., 42} and {105, 112, 119, ..., 196} has 14 elements. There are many ways in which a 1-1 correspondence can be set up between the two sets.

Answers to Exercises 5-6. Page 174.

1.	a.	996	d.	1,423	g.	903	j.	1660 R1
	b.	101	e.	561	h.	1,408,744		
	c.	5,073	f.	14,476	i.	79		

2. a. 55_{seven} b. 1101_{seven} c. 34_{seven} d. 121_{seven}

3. a. {1, 2, 3, 6}
   b. {1, 3}
   c. {1, 2, 3, 4, 6, 12}
   d. {1, 3, 13, 39}

4. a. {1_{seven}, 10_{seven}}
   b. {1_{seven}, 2_{seven}, 3 seven, 6_{seven}}
   c. {1_{seven}, 2_{seven}, 3 seven, 4_{seven}, 6_{seven}, 15_{seven}}
   d. {1_{seven}}

5. a. 3932
   b. 20,394
   c. 5,077
   d. 95,985
   e. 364
   f. 607 Remainder 12
   g. 6 Remainder 5

6. a. 6
   b. 28
   c. 12
   d. 1

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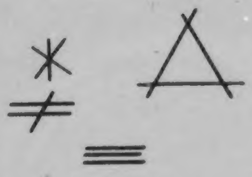
7. a. 15_{seven} b. 5_{seven} c. 11_{seven}

8. a. $\frac{3}{4}$
   b. $\frac{2}{3}$
   c. 20
   d. $\frac{1}{2}$
   e. $\frac{5}{21}$
   f. $\frac{1}{3}$

9. a. 1 or 3
   b. 2
   c. 0



10. a. 8 (each corner of the room)
    b. 12 (each "edge" of the room).

11. Divisible by 2: b. 101_{five} and d. 101_{seven}

12. Multiplies of 3: c. 14_{five}, d. 15_{seven} and e. 17_{eight}

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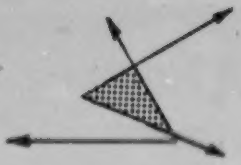
13. Composite Numbers
    a. 24_{five}
    d. 91
    e. 63
    h. 35_{nine}

    Prime Numbers
    b. 111_{two}
    c. 155_{seven}
    f. 10_{two}
    g. 103

14. a. one example:



b. one example:

15. a. seven hundred thousand three
b. eight hundred three thousand forty
c. six hundred ten thousand five hundred two
d. one hundred twenty-nine thousand forty-seven

16. a. $\{0, 1, 2, 3, 4\}$
b. 14
c. $\{1, 31\}$ 31 is a prime number. Hence the only possible factors are 31 and 1.

17. a. 801
b. 307,000
c. 21,024
d. 4,500,000

18. a. No; $2 + 7 = 9$. Only one example to the contrary is needed to show that this set is not closed with respect to an operation.
b. No; $7 \times 3 = 21$. The product of two numbers such that both are greater than 1 will always produce a composite number.

19. a. $<$
b. $<$
c. $<$
d. $a = 16$, $b = 12$, $g = 4$, $f = 2$.

20. a. $=$ or $<$
b. $=$ or $>$
c. $=$ or $<$
d. $=$ $>$ or $<$

21. a. Commutative Property of Multiplication
b. Associative Property and Commutative properties of multiplication.
c. Only one. (see Unique Factorization Property)

22. a. $\frac{1}{10}$ b. $\frac{1}{5}$ c. $\frac{1}{2}$ d. 1
(see discussion of exercises 5-6)

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23. a. 1 b. 26 c. 35 d. 41
(see discussion of exercises 5-6)

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24. a. Yes (see discussion of exercises 5-6)
    b. Yes
    c. Yes

5-7. Least Common Multiple (L.C.M.)

Skills and understandings to be developed:

1. To find common multiples of several numbers.
2. To find the least common multiple of several numbers in two ways--by listing the multiples of each and by using a complete factorization of each number.

It is recommended that the abbreviation (L.C.M.) not be used in this section. Use the complete phrase "least common multiple" in order to fix the meaning of the concept in the minds of the pupils.

The treatment of zero as a multiple must be done carefully. That zero is a multiple of any whole number is introduced here so that we may be consistent in the definition of an even number, i.e., "a number divisible by 2," or "a number ending in 0, 2, 4, 6, or 8." A careful treatment of zero should help clarify some of the special properties of zero.

That the "least common multiple" is the smallest counting number is an important distinction. Otherwise, for any members of a set of numbers, the least common multiple will be zero. Since zero as a least common multiple is not particularly useful to us in mathematics we choose to use the least counting number in a set of common multiples. Provide drill on finding the least common multiple by listing multiples. This provides some basic drill in multiplication and division. Problem 13 is designed to help students "discover" a shorter way to find the least common multiple. Do not insist on the use of this method at this time, but encourage the students to use it if they understand why it works.

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Method of finding the Least Common Multiple by Complete Factorization.

Example: Find the least common multiple of 12 and 18. First find a complete factorization of each number:

$12 = 2^2 \times 3$

$18 = 2 \times 3^2$.

Any number which is a multiple of 12 must have 2^2 as a factor and also 3 as a factor. Any number which is a multiple of 18 must have 2 as a factor and also 3^2 as a factor. Thus a number which is a common multiple of 12 and 18 must have among its factors all of the following:

$$2^2, \quad 3, \quad 2, \quad 3^2.$$

A number which has 2^2 as a factor certainly has 2 as a factor. Similarly, a number which has 3^2 as a factor certainly has 3 as a factor. Thus any number which has the two factors 2^2 and 3^2 will have all four of the factors 2^2, 3, 2, 3^2. The smallest number which has the factors 2^2, and 3^2 is $2^2 \times 3^2 = 36$.

On the other hand, $2^2 \times 3^2$ is a multiple of 12 (which is $2^2 \times 3$) and it is also a multiple of 18(which is 2×3^2). It is their least common multiple. Notice that the power of each prime factor in the least common multiple is the larger of the powers to which it occurs in the two given numbers.

Summary of Method: 1. Find a complete factorization of each number. 2. Notice which primes occur in at least one of the factorizations. 3. For each prime noticed in 2 , write the largest power of it which occurs in any of the factorizations. 4. Multiply all the powers written in 3; this product is the least common multiple.

The pupils will find it easier to pick out the primes and exponents in steps 2 and 3 above if the original numbers are written in a column with the complete factorization of each number to the right of the number.

Answers to Exercises 5-7 Page 180:

1. a. The set of multiples of 6 less than 100 is {0, 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96}.

   b. The set of multiples of 8 less than 100 is {0, 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96}

   c. The set of multiples of 9 less than 100 is {0, 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99}

   d. The set of multiples of 12 less than 100 is {0, 12, 24, 36, 48, 60, 72, 84, 96}

2. a. The set of common multiples of 6 and 8 less then 100 is {0, 24, 48, 72, 96}.

   b. The set of common multiples of 6 and 9 less than 100 is {0, 18, 36, 54, 72, 90}

   c. The set of common multiples of 6 and 12 less than 100 is {0, 12, 24, 36, 48, 60, 72, 84, 96}

   d. The set of common multiples of 8 and 9 less than 100 {0, 72}

   e. The set of common multiples of 8 and 12 less than 100 is {0, 24, 48, 72, 96}

   f. The set of common multiples of 9 and 12 less than 100 is {0, 36, 72}

3. a. The least common multiple of 6 and 8 is 24.

   b. The least common multiple of 6 and 9 is 18.

   c. The least common multiple of 6 and 12 is 12.

   d. The least common multiple of 8 and 9 is 72.

   e. The least common multiple of 8 and 12 is 24.

   f. The least common multiple of 9 and 12 is 36.

4. a. 10

   b. 12

   c. 30

   d. 12

   e. 30

   f. 60

   g. 42

   h. 72

5. a. 6
   b. 15
   c. 21
   d. 35
   e. 22
   f. 55
   g. 26
   h. 77
   i. 39
   j. 143
   k. 30
   l. 667

6. a. prime numbers
   b. The L.C.M. of two different prime numbers is equal to the product of the two numbers.

7. a. 12
   b. 8
   c. 20
   d. 18
   e. 40
   f. 60
   g. 60
   h. 60
   i. 30
   j. 24

8. Composite Numbers

9. a. Yes 8 is the L.C.M. of 4 and 8.
   b. No The L.C.M. of 8 and 9 is 72.

10. a. 6
    b. 29
    c. a

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11. a. 6
    b. 29
    c. a

12. a. No; what is the L.C.M. of 2 and 3?
    b. For two different prime numbers a and b, the L.C.M. is the product of the two numbers $a \cdot b$.
    c. For three different prime numbers, a, b, and c, the L.C.M. is the product of the three numbers, $a \cdot b \cdot c$.

13. (See the Discussion of Exercises 5-7).
    a. 48
    b. 112
    c. 45
    d. 70
    e. 144
    h. 360
    i. 660
    j. 720
    k. 1000
    l. 6480

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f. 60 m. 7038

g. 72

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14. a. No.
    b. No.
    c. No.

15. a. Yes.
    b. Yes.
    c. Yes.
    d. 0 .
    e. No. Zero is not a counting number.

Exercises 5-8.

The purpose of this section is to summarize in general the new material introduced in this chapter. Rather than produce a glossary, which might confuse pupils with definitions "out of context," we chose briefly to discuss the new terms. Attention is drawn to the terms which we wish to stress by writing them out in capital letters.

The diagram used in the introductory paragraph of this section might be discussed in class at some length to assure pupils' understanding of the relations among the sets of numbers included here. Pupils should understand, for example, that the set of prime numbers is a subset of the set of whole numbers and a subset of the counting numbers; similarly, for the set of composite numbers and the number "one." They should also understand that the number "zero" is a subset of the whole numbers but is not a subset of the counting numbers. Care should be used to explain these ideas. It is not intended that the word "subsets" be introduced, because the ideas involved in the diagram can be discussed without using set terminology.

Another purpose of this section is to fix in the pupils' minds the differences between the greatest common factor and the least common multiple. Note that we have avoided using the abbreviations L.C.F. and L.C.M. in the pupils' materials.

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It is recommended that these abbreviations not be used at this time.

The discussion should emphasize again that the greatest common factor of two different numbers is smaller than one or both of the numbers. Also, the least common multiple of two different numbers is greater than one or both of the numbers.

Answers to Exercise 5-8. Page 185.

1. a. 1
   b. 2
   c. 7
   d. 5
   e. 12
   f. 3
   g. 1
   h. 6
   i. 3
   j. 2
   k. 9
   l. 4
2. a. 6
   b. 24
   c. 14
   d. 75
   e. 36
   f. 105
   g. 989
   h. 858
   i. 663
   j. 5402
   k. 2520
   l. 31,372
3.

	Product of the numbers in problem 1		Product of the L.C.F. and L. C. M. of the numbers in problem 1
a.	2 × 3 = 6	a.	1 × 6 = 6
b.	6 × 8 = 48	b.	2 × 24 = 48
c.	7 × 14 = 98	c.	7 × 14 = 98
d.	15 × 25 = 375	d.	5 × 75 = 375
e.	12 × 36 = 432	e.	12 × 36 = 432
f.	15 × 21 = 315	f.	3 × 105 = 315
g.	23 × 43 = 989	g.	1 × 989 = 989
h.	66 × 78 = 5148	h.	6 × 858 = 5148
i.	39 × 51 = 1989	i.	3 × 663 = 1989
j.	74 × 146 = 10,804	j.	2 × 5402 = 10,804
k.	45 × 72 × 252 = 22,680	k.	9 × 2520 = 22,680
l.	44 × 92 × 124 = 501,952	l.	4 × 31,372 = 155,488

Note to teachers: It is always true that the product of two numbers is equal to the product of their G.C.F. and their L.C.M. This can be seen from the following example:

Let $r = 2^3 \times 5 \times 7$ and $s = 2 \times 5^2 \times 13$

To get the G.C.F. we take the product of the primes occuring in both raised to the smaller power: 2×5. To get the L.C.M. we take the product of the primes raised to the larger power:

$2^3 \times 5^2 \times 7 \times 13$.

Then $rs = 2^3 \times 5 \times 7 \times 2x5^2 \times 13$

and G.C.F. times L.C.M. $= 2 \times 5 \times 2^3 \times 5^2 \times 7 \times 13$.

One product is the same as the other except that the members are rearranged.

This is not true for three or more numbers. See part I.

4. a. {4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30}

   b. {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47}

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5. a. The product $a \cdot b$

   example (1) The G.C.F. of 3 and 5 is 1.
   The L.C.M. of 3 and 5 is 15.

   example (2) The G.C.F. of 4 and 9 is 1.
   The L.C.M. of 4 and 9 is 36.

   b. No. Not necessarily.
   The G.C.F. of 2, 3 and 4 is 1.
   The L.C.M. of 2, 3, and 4 is $2^2 \cdot 3$, or 12. It is not $2 \cdot 3 \cdot 4$.

6. a. 2 is a prime number. It is the only even prime number.

   b. All primes except 2 are odd, i.e., 3, 5, ...,.

   c. One. Only the prime number 5 has an ending in 5. All other numbers ending in 5 are multiples of 5, i. e., 15, 25, 35, ...,.

   d. 2 and 5.

   e. 1, 3, 7, and 9.

7. The two numbers are the same. For example: The L.C.M. of 7 and 7 is 7. The L.C.M. of 7 and 7 is 7.

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8. a. 1. One is the least common factor of any two whole numbers.

b. No answer is possible for the greatest common multiple of any two whole numbers.

9. a. (Bulbs and rows may be interchanged.)

Rows	Bulbs per row
1	112
2	56
4	28
8	14
16	7

10. a. 3, 6, 9, 12, 15--first bell,
5, 10, 15--second bell.
They strike together again in 15 minutes.

b. 6, 12, 18, 24, 30--first bell,
15, 30--second bell.
They strike together again in 30 minutes.

c. 15 is the least common multiple of 3 and 5.
30 is the least common multiple of 6 and 15.

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11. a. Yes. The G.C.F. of 6 and 6 is 6.
The L.C.M. of 6 and 6 is 6.

b. No. The G.C.F. of the members of a set of numbers can never be greater than the largest number of the set of numbers because a factor of a number is always less than a multiple of the number unless the multiple is zero.

c. No. The L.C.M. of the members of a set of numbers can never be less than the largest member of the set of numbers.

The least common multiple of two numbers is at least as big as the larger of the two numbers (since the L.C.M. is a multiple of the larger number). The greatest common factor of two numbers is no larger than the smaller of the two numbers (since the G.C.F. is a factor of the smaller number). If the least common multiple and greatest common factor are equal, the larger and smaller number must also be equal.

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Illustrate with 4 and 6, locating the L.C.M. and G.C.F. on the number line; then use 4 and 4.

12. a. No. It is not possible to have exactly four numbers between two odd numbers. Between any two odd primes there is always an odd number of numbers. If they are consecutive odd primes all the numbers between would have to be composite.

b. Yes. For example, between 23 and 29 there are exactly 5 composite numbers; 24, 25, 26, 27, 28.

13. a. 135, 222, 783, and 1065 are all divisible by three.

b. 222 is the only number divisible by six

c. 135 and 783 are divisible by nine.

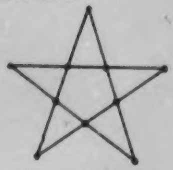
d. 135 and 1065 are divisible by five.

e. 135 and 1065 are divisible by fifteen.

f. None of the numbers are divisible by four.

14. Many different answers should be expected from the students; in a way his reasons for his answer are more important than the answer itself. The most obvious answer is that as far as multiplication is concerned, the prime numbers are the building blocks of all counting numbers, since any counting number can be expressed as a product of prime numbers. If we know a complete factorization of a number, then we can find all its factors. Also from complete factorizations, the G.C.F. and L.C.M. of sets of numbers may also be found.

15. BRAINBUSTER. The pattern is a five-pointed star.



16. There is no greatest prime number.

The following proof is for the information of the teacher. It is not expected that any except the most brilliant students would profit by it in the 7th grade.

To show there is no largest prime number, we will show that if p is any prime, there is another prime larger than p. Denote by M the product of all the primes less than or equal to p: $M = 2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times p$.

Then M + 1 is certainly larger than p and M + 1 has at least one prime factor (it may be a prime). But M + 1 does not have any of the primes 2, 3, 5, 7, ..., p as a factor since division by any of these primes leaves a remainder of 1. Thus all the prime factors of M + 1 are larger than p, and hence p is not the largest prime. Since p was an arbitrary prime, there is no largest prime.

Sample Questions for Chapter 5.

This set of questions is not intended as a chapter test. Teachers should construct a chapter test carefully by combining selected items from this set of questions and questions of their own writing. Care should be exercised to avoid making the test too long.

I. True-False Questions

(T) 1. Every composite number can be factored into prime numbers in exactly one way, except for order.

(F) 2. The sum of an odd and an even number is always an even number.

(T) 3. Some odd numbers are not primes.

(F) 4. Every composite number has only two prime factors.

(F) 5. The number 51 is a prime.

(T) 6. All even numbers have the factor 2.

(F) 7. Any multiple of a prime number is a prime.

(T) 8. 8 can be expressed as the sum of twin primes.

(T) 9. Even though 1 has as factors only itself and 1, it is not considered a prime number.

(F) 10. All odd numbers have the factor 3.

(F) 11. No even number is a prime.

(T) 12. The least common multiple of 3, 4, and 12 is 12.

(F) 13. The number one is a prime factor of all counting numbers.

(F) 14. The greatest common multiple of 2, 5, and 10 is 100.

(T) 15. The difference between any two prime numbers greater than 100 is always an even number.

(F) 16. The number 41 is a composite number.

(F) 17. A multiple of 6 must be a multiple of 18.

(F) 18. The least common multiple of 2 and 6 is 12.

(T) 19. The greatest common factor of any two even numbers is at least 2.

(T) 20. The difference between any two prime numbers greater than 8 is always an even number.

(T) 21. A fraction can be reduced if the greatest common factor of its numerator and denominator is greater than one.

(F) 22. If z is different from 0 then $\frac{3z}{z} = 2z$

(F) 23. A composite number is one whose only factors are itself and one.

(T) 24. The multiplier and multiplicand are both called factors of the product.

(T) 25. There is only one even prime number.

(F) 26. The least common multiple of 3, 4, 6, and 12 is 24.

II. Multiple Choice

27. Which one of the following is an odd number?
    a. $(24501)_{\text{six}}$
    b. $(60654)_{\text{eight}}$
    c. $(\text{TE}104)_{\text{twelve}}$
    d. $(110100)_{\text{two}}$
    e. None of the above 27. a.

28. The greatest common factor of 48 and 60 is:
    a. 2×3
    b. $2 \times 2 \times 3$
    c. $2 \times 2 \times 2 \times 2 \times 3 \times 5$
    d. $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5$
    e. None of the above 28. b.

29. Every counting number has at least the following factors:
    a. Zero and one
    b. Zero and itself
    c. One and itself
    d. Itself and two
    e. None of the above 29. c.

30. In a complete factorization of a number
    a. All the factors are primes.
    b. All the factors are composites.
    c. All the factors are composite except for the factor 1.
    d. All the factors are prime except for the factor 1.
    e. None of the above 30. a.

31. How many different prime factors does the number 72 have?
    a. 0
    b. 1
    c. 2
    d. 3
    e. None of the above 31. c.

32. The least common multiple of 8, 12, and 20 is:
    a. 2×2
    b. $2 \times 3 \times 5$
    c. $2 \times 2 \times 2 \times 3 \times 5$
    d. $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 5$
    e. None of the above 32. c.

33. Which of the following is an even number?
    a. $(100)_{\text{three}}$
    b. $(100)_{\text{five}}$
    c. $(100)_{\text{seven}}$
    d. $(100)_{\text{eleven}}$
    e. None of the above 33. e.

34. Which of the following numbers is odd?
    a. 17 × 18
    b. 18 × 11
    c. 11 × 20
    d. 99 × 77

e. None of the above 34. d.

35. Which of the following is not a prime number? There is only one.

a. 271
b. 277
c. 281
d. 282
e. 283 35. d.

36. Let a represent an odd number, and b represent an even number; then a + b must represent

a. an even number.
b. a prime number.
c. an odd number.
d. a composite number.
e. none of these. 36. c.

37. If n represents an odd number, the next odd number can be represented by

a. n + 1
b. n + 2
c. n + 3
d. 2 × n
e. None of these. 37. b.

38. A counting number is an even number if it has the factor:

a. 5
b. 3
c. 2
d. 1
e. None of these 38. c.

39. Which of the following sets contains only even numbers?

a. {2, 3, 4, 5, 9, 10}
b. {2, 5, 10}
c. {3, 5, 9}
d. {2, 4, 10}
e. {3, 9} 39. d.

40. Which of the following is a prime number?
a. 4
b. 7
c. 9
d. 93
e. None of these 40. b.

41. Which of the following is not a factor of 24?
a. 2
b. 3
c. 4
d. 9
e. 12 41. d.

42. Which of the following is the complete factorization of 36?
a. 4 × 9
b. 2 × 3 × 6
c. 3 × 12
d. 2 × 18
e. 2 × 2 × 3 × 3 42. e.

43. The numbers 8, 9, 16, 20, 27, and 72 are all
a. prime numbers.
b. even numbers.
c. odd numbers.
d. composite numbers
e. none of these. 43. d.

44. How many multiples of 4 are there between 25 and 50?
a. 5
b. 7
c. 9
d. 11
e. None of these 44. e.

45. If a whole number has 6 as a factor, then it also has the following factors:
a. 2 and 3
b. 2 + 3
c. 12
d. all multiples of 6
e. none of these 45. a.

46. Suppose p and q are counting numbers and q is a factor of p; then:
    a. q is a multiple of p
    b. p is a multiple of q
    c. q must be a prime number
    d. the greatest common factor of p and q must be less than q
    e. none of these 46. b.

47. The greatest common factor of 60 and 42 is
    a. 2 × 3
    b. 2 × 2 × 3
    c. 2 × 3 × 5
    d. 2 × 3 × 7
    e. none of these 47. a.

48. The greatest common factor of two different numbers must be:
    a. a composite number
    b. a prime number
    c. smaller than one of the numbers
    d. smaller than both of the numbers
    e. none of these 48. c.

49. The least common multiple of two numbers is always:
    a. their product
    b. the product of their factors
    c. the sum of their factors
    d. the sum of the numbers
    e. none of these 49. e.

50. Which of the following statements describes a prime number?
    a. a number which is a factor of a counting number
    b. a number which has no factors
    c. a number which does not have 2 as a factor
    d. a number which has exactly 2 different factors
    e. none of these 50. d.

51. How many prime numbers are there between 20 and 40?
    a. 4
    b. 5
    c. 8

d. 9
e. none of these 51. a.

52. When two prime numbers are added the sum is
a. always an odd number
b. always an even number
c. always a composite number
d. always a prime number
e. none of these 52. e.

53. The set of factors of the number 12 is
a. {1, 2, 3, 4, 8, 12}
b. {1, 2, 3, 4, 6, 12}
c. {1, 2, 3, 4, 6}
d. {2, 3, 4, 6, 12}
e. none of these 53. b.

54. How many different factorizations of two factors each does 175 have?
a. 2
b. 3
c. 4
d. 5
e. none of these 54. b.

55. The number of factors in a complete factorization of 182 is
a. 2
b. 3
c. 4
d. 5
e. none of these 55. b.

56. Which one of the following is not a prime number?
a. 17
b. 23
c. 47
d. 49
e. none of these 56. d.

57. Which one of the following numbers has an odd number as a prime factor?
a. 8

b. 24
c. 32
d. 128
e. All of these 57. b.

58. The greatest common factor for 51, 68, and 17 is . . .
a. 1
b. 3
c. 7
d. 17
e. None of these 58. d.

59. The greatest common factor for 72 and 36 is . . .
a. 6
b. 12
c. 2×3^2
d. $2^2 \times 3^2$
e. none of these. 59. d.

60. What is the greatest common factor of 48 and 64?
a. 2^3
b. $2^2 \times 3$
c. $2^3 \times 3^2$
d. 2×3^2
e. none of these. 60. e.

61. The numbers 9, 25, and 36 are all . . .
a. prime numbers
b. even numbers
c. odd numbers
d. composite numbers
e. none of these. 61. d.

III. Problems Answers

62. Find a complete factorization of each number
a. 16 62. a. 2^4
b. 100 b. $2^2 \times 5^2$
c. 57 c. 3×19

63. Find the greatest common factor of each set of numbers.
a. 5 and 25 63. a. 5

b. 18 and 27 b. 9
c. 60, 36, and 24 c. 12

64. Find the least common multiple of each set of numbers.
a. 6 and 8 64. a. 24
b. 7 and 9 b. 63
c. 16, 12 and 20 c. 240

65. Name the numbers in the set of counting numbers between 16 and 25 which have a factor of 3. 65. {18, 21, 24}

66. What is the sum of all of the factors of 28 less than 28?
66. 28.

67. Find the smallest number which has a factorization composed of 3 composite numbers. 67. $64 = 4^3$

68. Show that a product is even if one (or more) of its factors is even
68. Every number which has 2 as one of its factors must be divisible by 2.

69. Is the set of even numbers closed under addition? Show that your answer is correct. 69. Yes

70. If the complete factorization of a number is $2 \times 3 \times 5 \times 7$, what factors less than 20 does the number have?
70. 1, 2, 3, 5, 6, 7, 10, 14, 15.

71. Find all the common multiples less than 100 of these three numbers: 3, 6, 9. 71. 18, 36, 54, 72, 90

72. Write all the factorizations of two factors for the number 210.
72. 1×210, 2×105, 3×70, 5×42, 6×35
7×30, 10×21, 14×15.

Chapter 6

THE RATIONAL NUMBER SYSTEM

Overview

The most important objective of this chapter is to present the reasons behind the rules for the arithmetic operations of multiplication, division, addition, and subtraction on the set of rational numbers. The student will be able to use these rules correctly and with facility if he fully understands the reasons behind them. We reiterate the principle that mathematics stresses reason over rote learning and concept over technique. At the same time, an equally important goal at this level is to reinforce techniques which the pupil has already learned. These two goals should not be incompatible. We are assuming that the pupils have already been introduced to these operations and have had some practice in them in an earlier grade. This means that they already know what a fraction is--at least in easy situations.

The seventh grader is probably not ready for a postulational approach to the study of number systems. The whole numbers and the rational numbers are part of nature for him. To speak of constructing the rational number system from the whole numbers would be meaningless at his level, as it would be to speak of the problem of defining the operations in this set of numbers. For this reason we have not stated postulates or definitions. Instead of posing the problem of how to define multiplication of rational numbers, for instance, and then building toward this definition, we have proposed to the pupil that we examine the reasons why multiplication behaves as it does. We try to motivate the usual rule for multiplication, and then state this rule clearly and concisely. Instead of treating the commutative, associative, and distributive properties as postulates we simply point them out and state them clearly early in the discussion of the operations and then use them in subsequent discussion. It is hoped that as he follows this treatment the pupil will begin to see the skeleton of a mature mathematical approach to number systems without being smothered by the more subtle ideas which are necessary for complete understanding. Do not tell your pupils, however, that they are really just talking in a very naive way about something that is really somewhat sophisticated.

The student will probably bring a wealth of terminology and calculating procedures with him from the sixth grade. He is probably familiar with the terms mixed number, proper fraction, improper fraction, decimal fraction, and lowest terms. We suggest that you use the terms that are familiar to you and to your students. We have made no attempt to build such a vocabulary in this chapter because these terms are not used extensively in subsequent mathematics.

We wish to discourage the indiscriminate use of the operations of "cancelling" and "reducing fractions". Instead we stress the operations of "factoring" and "finding the simplest form for a fraction". Thus instead of

$$\frac{12}{15} \div \frac{3}{5} = \frac{\overset{4}{\cancel{12}}}{\underset{\cancel{5}}{\cancel{15}}} \cdot \frac{\cancel{5}}{3} = \frac{4}{3}$$

we prefer

$$\frac{12}{15} \div \frac{3}{5} = \frac{12}{15} \cdot \frac{5}{3} = \frac{4 \cdot 3}{5 \cdot 3} \cdot \frac{5}{3} = \frac{4 \cdot 3 \cdot 5}{5 \cdot 3 \cdot 3} = \frac{3}{3} \cdot \frac{5}{5} \cdot \frac{4}{3} =$$

$$1 \cdot 1 \cdot \frac{4}{3} = \frac{4}{3}\,.$$

The term "factor" has been used in an earlier chapter. Moreover in ninth grade algebra a student is asked to factor $x^2 - 1$.

We use freely the terminology and concepts developed in earlier chapters. Thus the terms factor, prime, greatest common factor, least common multiple, ray, and segment are used.

In order to discuss the arithmetic operations intelligently, indeed to tell what the elements of our system of rational numbers are, we must first discuss the meaning of a "fraction" and the "equality of fractions". We must point out that a fraction is a name for a number and that a number has many names. We shall not belabor the difference between a number and its name.

Various means are used to describe a rational number--measures of quantities, points on the number line, and quotients of whole numbers. We hope that you will find that we have introduced the number sentence $b \cdot x = a$ in an easy way. We realize that students will vary greatly in the amount of abstract mathematics they will comprehend. These different approaches can be viewed as three distinct explanations, or each can be viewed as an interpretation of the other. This chapter can serve as a bridge from arithmetic to algebra. Our concurrent use of the number line and of equations or number sentences is intermediate between a concrete interpretation of number and the abstract manipulation of number as represented by a symbol in an equation. You may want to point out that while piles of sand or points on the line are not rational numbers, problems involving them provide motivation for the abstract mathematical definitions which create the system of rational numbers. You may also want to give more "social-type" exercises and examples.

All pupils should be able to see that $\frac{a}{b}$ represents the number x such that $b \cdot x = a$ and to know that a rational number is one which can be written as a quotient of a whole number and a counting number.

In connection with our use of the number sentence, you will note that the most elaborate equation discussed is of the form $ax = b$ or $a + x = b$. It is always the case that x is to stand for a number. We are usually seeking a replacement for it, either from the set of whole numbers, or from the set of rational numbers. The symbol "x" is not to be regarded as a variable. In any number sentence or equation, the point is made that we have a representation of the same number in two different ways. When an operation is performed on this number, we get the same result, no matter which representation is used. Thus when an operation is performed on both sides of an equality, the equality remains true, although the numbers involved may be changed.

You may have other highly successful teaching methods for presenting the arithmetic operations with rational numbers. You may even have found that equations cannot be introduced in the seventh grade. We hope you will try to understand the approach we are making here. By simply regarding $\frac{a}{b}$ as the replacement for x which makes $bx = a$ true, we think we are able to present sound mathematical reasons behind the arithmetic operations. This is actually the point of view from which fractions are introduced traditionally. The number $\frac{5}{7}$ is what you get when you divide 5 into 7 equal parts; this is another way of saying that if you take seven of these parts you get 5. It is somewhat analogous to answering the question: What was the color of Washington's white horse? This way of looking at fractions may be a little difficult for the students at the time, especially if the teacher has not tried it before. But it should pay dividends in the long run. One such place is in the introduction of negative numbers from an analogous point of view.

6-1. History of Fractions

The historical remarks show that arithmetic concepts were introduced with some difficulty into ancient civilizations. Egyptian and Babylonian fractions provide an opportunity for a brief review of the operations you think the students should already be able to perform.

6-2. Rational Numbers

Two ideas, on which the rest of the chapter depends, are introduced in this section.

1. Equivalent fractions are merely different names for the same number.
2. $x = \frac{a}{b}$ because $bx = a$, $b \neq 0$.

It is felt that the students are well grounded at this stage in considering fractions by means of specific examples, and in realizing that $\frac{3}{4}$ is either $\frac{1}{4}$ of 3 or 3 times $\frac{1}{4}$, but that they have had little experience in considering a fraction as the result of division, or in division as an inverse operation to multiplication.

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The chapter has been planned to give more practice in these concepts, since they are basic as one continues with mathematics. Other approaches are easier, but they do not achieve this result. It must be emphasized that the rational number $\frac{a}{b}$ is the number whose product with b is a, and that a concise way of saying this is: $\frac{a}{b}$ is the number x for which $bx = a$.

In this section the switch from the arithemtic symbols for multiplication and division to those generally used in science and mathematics, is made. The symbol 5×6 is now written as $5 \cdot 6$ or $5(6)$, $a \times b$ as either $a \cdot b$ or ab, and $5 \div 6$ as $\frac{5}{6}$.

Some special comment on the use of the words "fraction" and "rational number" seems in order here. In the text we have tried to be very careful to use the word "fraction" only when we refer to a symbol of a certain kind. It is a name for a number. There are several reasons for this. In the first place, we want to refer to the numerator and denominator which are parts of a symbol for a number. Thus we can speak of the numerator of a fraction but not the numerator of a number. This reinforces the important idea that a number has many names — some of these names are fractions. Secondly, we are looking ahead to the time when we shall want to call such things as:

$$\frac{\sqrt{2}}{2} \quad \text{and} \quad \frac{ax + b}{cx + d}$$

fractions.

Thus we write of equal numbers but not of equal fractions. When we write $\frac{2}{3} = \frac{4}{6}$ we do not mean that the fractions are equal but merely that the numbers which they represent are equal. Fractions which represent the same numbers we call "equivalent" but we usually can avoid this word. Also, we add and subtract numbers but we do not add and subtract fractions.

Making these distinctions requires great care at times since traditionally the word "fraction" is used in both senses, but it is felt that such care will avoid confusion now and later. Though it is very important that the text be precise in this respect, it

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is probably too much to expect pupils at this stage to exercise equal care in their expression. The teacher will know where careful usage will avoid confusion (and there are many such places) and where use of "fraction" in both senses will result in no difficulty.

Answers to Exercises 6-2.

1. (a) 1, 2, 3, etc.
   (b) 0, 1, 2, 4, etc.
   (c) 0
   (d) Numbers such as $\frac{5}{3}$, $\frac{4}{11}$
2. All are rational numbers.
3. (a) 6
   (b) 3
   (c) 5
   (d) 4
   (e) 1
   (f) $6x = 18$
   (g) $3x = 4$
   There are other correct answers for f and g.
4. (a) 2 (b) 3 (e) 7 (f) 3
5. (a) $5 \cdot x = 1$
   (b) $4x = 4$
   (c) $3x = 11$
   (d) $9x = 63$
   (e) $5x = 0$
   (f) $11x \doteq 123$
   There are other correct forms for the answers.
6. (b) 1 (d) 7 (e) 0
7. (a) true
   (b) false
   (c) true
   (d) true
   (e) true

8. (a) If x is the number of cookies each boy should receive, then $3x = 12$.

   (b) If x is the number of miles Mr. Carter drove on each gallon of gasoline, then $10x = 160$.

   (c) If x is the number of bags of cement needed for each foot of walk, then $30x = 20$.

   (d) If x is the number of pupils in each group, then $4x = 32$.

   (e) If x is the number of sheets of paper each pupil receives, then $24x = 12$.

6-3. Properties of Rational Numbers

Here it is pointed out by examples that the closure, commutative, associative, and identity properties of addition and multiplication, and the distributive property of multiplication over addition, hold for the rational numbers as well as for the whole numbers. Many teachers may find this an opportune review.

On the basis of these properties, it is shown that $\frac{3}{2}$ and $\frac{5\cdot 3}{5\cdot 2}$ represent the same number by considering a corresponding problem in terms of multiplication. It is not appropriate at this time to show $\frac{3}{2} = \frac{5\cdot 3}{5\cdot 2}$ by saying $\frac{3}{2} = \frac{5}{5} \cdot \frac{3}{2}$ because no method for multiplying fractions has been discussed.

By generalizing from this example, Property 1 is obtained. For selected students you may wish to give an argument for Property 1 as follows:

Let us now look at the general situation. If a, b, and k are whole numbers, b and k not zero, and $bx = a$, then $x = \frac{a}{b}$. Since bx and a are names for the same number, $k(bx)$, $(kb)x$, and ka are different names for one number.

$$(kb)x = ka$$

If $(kb)x = ka$, then $x = \frac{ka}{kb}$.

The simplest fraction for a rational number is a fraction in which the numerator and denominator have no common factor except 1. We have continued to call this "the simplest form of the fraction" although this presents logical difficulties. By this definition the simplest form for $\frac{10}{5}$ is $\frac{2}{1}$. When this occurs, pupils should be encouraged to also write the number as 2.

Until now pupils may have been urged to always change improper fractions to mixed numbers. This is unrealistic. Both forms are correct and both forms have their uses. In many exercises there are various correct forms for the answer. If a student gives an alternative correct form, he should receive full credit.

Answers to Exercises 6-3

1. All except k.
2. Correct ways are numerous. Some possibilities are:
   - (a) $\frac{2}{4}$, $\frac{3}{6}$, .5
   - (b) $\frac{4}{10}$, $\frac{6}{15}$, .4
   - (c) $\frac{1}{1}$, 1, $\frac{5}{5}$
   - (d) $\frac{6}{2}$, $\frac{3}{1}$, 3
   - (e) 0, $2 \cdot 0$, $3 \cdot 0$
   - (f) $\frac{2}{1}$, 2, $\frac{4}{2}$
   - (g) 0, $\frac{0}{1}$, $\frac{0}{2}$
   - (h) $\frac{5}{6}$, $\frac{10}{12}$, $\frac{50}{60}$
   - (i) $\frac{6}{5}$, $1\frac{1}{5}$, 1.2
   - (j) $\frac{6}{1}$, $\frac{12}{2}$, $\frac{18}{3}$
3.
   - (a) $\frac{1}{3}$
   - (b) $\frac{4}{5}$
   - (c) $\frac{15}{4}$
   - (d) $\frac{2}{3}$
   - (e) $\frac{4}{1}$ or 4
   - (f) $\frac{3}{1}$ or 3
   - (g) $\frac{1}{3}$
   - (h) $\frac{4}{5}$
   - (i) $\frac{5}{4}$
   - (j) $\frac{1}{3}$

4. (a) Commutative for multiplication
   (b) Associative for multiplication
   (c) Distributive
   (d) Distributive
   (e) Identity for multiplication
   (f) Identity for multiplication
   (g) Identity for addition
   (h) Commutative for addition
   (i) Associative for addition
   (j) Associative for multiplication

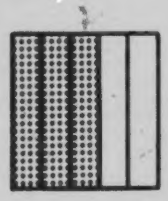
5. (a) 7 (c) 10
   (b) 8 (d) 3

6. (a) 75 cents, or $0.75, or $\frac{3}{4}$ of a dollar

   (b) $\frac{3}{4}$

7. Numerically there is no difference.

8. (a)



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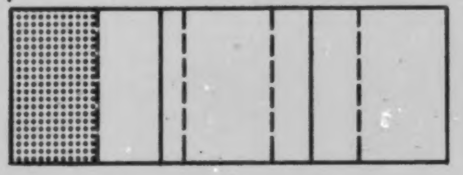

(c) $\frac{3}{5}$ and $\frac{1}{5}$

(d) $3 \cdot \frac{1}{5} = \frac{1}{5} \cdot 3$

9. $\frac{6}{8}$ or $\frac{3}{4}$

10. No difference in area.

11.

n	d
2	3
4	6
6	9
8	12
10	15

etc.

$\frac{n}{d} = \frac{2}{3}$

12. If 2x and 3 are names for the same number then

$$\begin{aligned}\tfrac{1}{2}(2x) &= \tfrac{1}{2} \cdot 3\\(\tfrac{1}{2} \cdot 2)x &= \tfrac{1}{2} \cdot 3\\1 \cdot x &= \tfrac{1}{2} \cdot 3\\x &= \tfrac{1}{2} \cdot 3\end{aligned}$$

But if $2x = 3$ then $x = \frac{3}{2}$. We have three names for the same number x, $\frac{3}{2}$, and $\frac{1}{2} \cdot 3$. Since $\frac{3}{2}$ and $\frac{1}{2} \cdot 3$ are names for the same number, $\frac{3}{2} = \frac{1}{2} \cdot 3$.

6-4. Reciprocals

If the product of two rational numbers is 1, these numbers are called reciprocals of each other. The topic of reciprocals is introduced relatively early in the chapter to provide for an application of the definition of rational numbers, additional review of skills learned in earlier grades, and in preparation for later topics, particularly division.

It is assumed that students already know that $n \cdot \frac{1}{n} = 1$, for $n \neq 0$. Many of them probably would also know that the number by which $\frac{a}{b}$ would be multiplied to obtain 1 is $\frac{b}{a}$. Nevertheless it should be profitable for them to review multiplication in these special cases in a setting which emphasizes the definition of a rational number. Such review should make some of the later developments much more meaningful.

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At this point it seems undesirable to think of the product $\frac{5}{6} \cdot \frac{6}{5}$ as $\frac{5}{5} \cdot \frac{6}{6}$ since such a step involves exchanging the numerator of the numerals for the rational numbers rather than using the rational number as a number on which operations can be performed without regard to the particular numeral used to express it.

Answers to Exercises 6-4

1. (a) 1
   (b) 1
   (c) 1
   (d) 1
   (e) 1
   (f) $\frac{92}{51}$
   (g) $\frac{7}{4}$
   (h) $\frac{12}{5}$
   (i) 1
   (j) 1
2. (a) $\frac{1}{11}$
   (b) $\frac{1}{201}$
   (c) $\frac{1}{47}$
   (d) 5
   (e) $\frac{7}{2}$
   (f) $\frac{3}{50}$
   (g) $\frac{7}{1000}$
   (h) $\frac{175}{346}$
3. (a) $\frac{1}{m}$
   (b) $\frac{1}{s}$
   (c) c
   (d) $\frac{s}{r}$
   (e) $\frac{w}{t}$
4. (a) $\frac{5}{3}$
   (b) $\frac{2}{7}$
   (c) $\frac{19}{15}$
   (d) $\frac{1}{2}$
   (e) $\frac{k}{r}$

5. (a) $7n = 8, \quad n = \frac{8}{7}$

(b) $20n = 15, \quad n = \frac{3}{4}$

(c) $17n = 100, \quad n = \frac{100}{17}$

(d) $11n = 2, \quad n = \frac{2}{11}$

(e) $13n = 1492, \quad n = \frac{1492}{13}$

(f) $300n = 6, \quad n = \frac{1}{50}$

(g) $475n = 5, \quad n = \frac{1}{95}$

(h) $36n = 64, \quad n = \frac{16}{9}$

6. Set of reciprocals is $\{1, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}\}$.

7. (a) If the number is between 0 and 1, the reciprocal is greater.

(b) If the number is greater then 1, the reciprocal is less.

(c) The reciprocal of 1 is 1.

(d) One of these sentences, and only one, is true for every counting number.

8. $\frac{2}{19}$

9. (a) $\frac{9}{2}$ (b) $\frac{2}{9}$

(c) They are reciprocals of each other.

10. (a) $\frac{15}{8}$ (b) $\frac{8}{15}$

(c) They are reciprocals of each other.

*11. (a) If $14n$ and 5 are names of the same number, then $\frac{1}{5}(14n)$ and $\frac{1}{5}(5)$ are names of the same number.

*12. $x = \frac{b}{a}$ and $y = \frac{a}{b}$

$$\frac{b}{a} \cdot \frac{a}{b} = \frac{b}{a} \cdot a \cdot \frac{1}{b} = b \cdot \frac{1}{b} = 1$$

Or in a more sophisticated way,

$$ax = b \qquad by = a$$

$$(ax)(by) = b \cdot a$$

$$(ab)(xy) = ab$$

$$xy = \frac{ab}{ab} = 1$$

The students many wish to think of $xy = w$, to have

$$(ab)(xy) = ab$$

$$(ab)w = ab$$

$$w = \frac{ab}{ab} = 1$$

$$xy = 1$$

6-5. Using the Number Line

Here the purpose of the introduction of the number line is partly for review and partly to discuss some of the previous parts of this chapter from a somewhat geometrical point of view. It was felt that the use of the figure should contribute to the understanding of some of the relationships involved. Most of the concepts dealt with here are already somewhat familiar to the pupils and the stress which the teacher lays on this section should depend partly on how familiar pupils are with these ideas. The chief new idea in this section concerns two methods of determining whether or not two given fractions represent the same number. This also looks forward to the discussion of the ordering of rational numbers in the last section of this chapter.

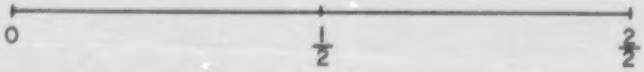
Many teachers will notice the absence of the terms "mixed number" and "improper fraction". The former was omitted in order to avoid some semantic difficulties since the term "mixed number" really refers to a property of the numeral rather than the number;

e.g., $\frac{21}{5}$ is not a mixed number in the usual sense even though $4\frac{1}{5}$ is. The adjective "mixed" refers merely to the way the numeral is written. The term "improper fraction" does not have this difficulty since it can be described as a rational fraction whose numerator is greater than its denominator. But there still did not seem to be any point in introducing it in this section. It is referred to in the last section of the chapter but not especially emphasized.

Example 4 comes very close to another method of determining whether or not two fractions represent the same number. Here, to compare $\frac{a}{b}$ and $\frac{c}{d}$ we write $\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{cb}{db}$. This shows immediately that $\frac{a}{b}$ and $\frac{c}{d}$ represent the same number if and only if $ad = cb$. This was not mentioned in the students' material because it was felt that unless the student discovered this for himself, the test would be a purely mechanical one. This also applies in the section on ordering.

Answers to Exercises 6-5.

1. (a)



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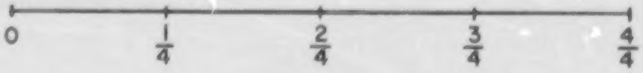

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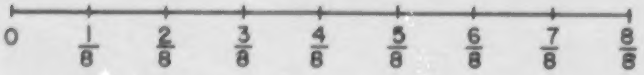

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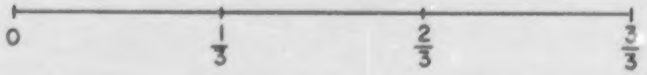

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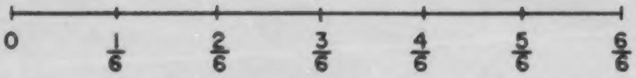

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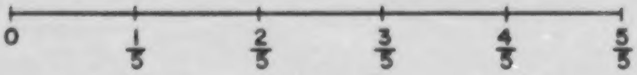
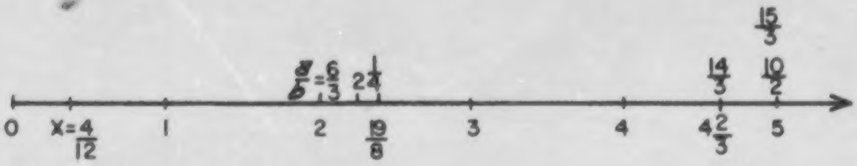


2.



3. (a) A: $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$

B: $\frac{1}{4} = \frac{2}{8} = \frac{3}{12}$

C: $\frac{3}{4} = \frac{6}{8} = \frac{9}{12}$

D: $\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$

E: $\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$

F: $\frac{1}{1} = \frac{2}{2} = \frac{3}{3}$

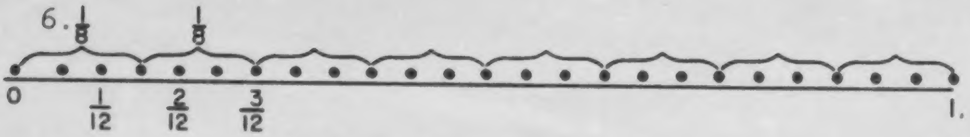
(b) The rational number located at point B is less than the rational number located at point A. Because point B is to the left of point A on the number line.

(c) Point A is to the left of point C. Therefore the rational number located at C is greater than the one located at A.

4. Here there are at least two interpretations for each. Perhaps the teacher would prefer to require two answers for each. For example in (a) one could exhibit a number line with 20 divisions and note that if it is divided into 5 equal parts, each will be 4 units long. Also, the same divisions could be labelled as fifths and then the line would be four units long. It is important that the students see both.

5. (a) $\frac{57}{39} = 1\frac{18}{39}$ (b) $\frac{137}{23} = 5\frac{22}{23}$

6.



7. (a) $\frac{6}{9} = \frac{24}{36}$ and $\frac{8}{12} = \frac{24}{36}$.

Hence the fractions are equal.

(b) $\frac{9}{15} = \frac{45}{75}$ and $\frac{20}{25} = \frac{60}{75}$.

Hence the fractions are not equal.

8. (a) Here one measures three units to the right from the 0 point, then moves two units to the left from the point 3, arriving at the point 1 which is equal to 3 - 2.

(b) Similar

Notice that here negative numbers are not involved. It is merely the idea that if we move to the right to add, we move in the opposite direction to subtract. The teacher should not belabor this point.

*9. One can certainly find whether two fractions are equal by changing them to fractions with equal numerators and examining their denominators. There are times when this method is easier than the other method, for instance, in examples like: Is $\frac{3}{11} = \frac{3}{10}$? Is $\frac{3}{11} = \frac{6}{23}$?

6-6. Multiplication of Rational Numbers

Using familiar products such as $2 \cdot \frac{1}{3}$ as a basis, a development through products such as $2 \cdot \frac{2}{7}$ leads to the general situation $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$. As before, the basic properties are used to develop the usual rule for the operation. Pupils should not be expected to reproduce such a development, but only to see how it leads to the procedure they already know.

You may prefer to place more emphasis on the properties of 1 and use these in place of Property 1, now that multiplication for rational numbers in fractional form is defined.

Further exercises may be obtained by asking that the answers in some of Exercises 6-6 be in a different form.

__Answers to Exercises 6-6__

1. (a) $\frac{1}{6}$
   (b) $\frac{2}{15}$
   (c) $\frac{3}{20}$
   (d) $\frac{6}{25}$
   (e) $\frac{3}{100}$
   (f) $\frac{3}{250}$
   (g) 1
   (h) $\frac{4}{9}$
   (i) $\frac{4}{9}$
   (j) $\frac{4}{3}$
   (k) $\frac{1}{9}$
   (ℓ) 6
   (m) 6
   (n) $\frac{1}{24}$
   (o) $\frac{1}{1000}$
   (p) $\frac{1}{10}$
   (q) $\frac{1}{10,000}$
   (r) $\frac{1}{1000}$
   (s) 10
   (t) 20
   (u) $\frac{1}{6}$
   (v) 1
   (w) $\frac{21}{4}$
   (x) $\frac{1}{15}$

2. (a) $\frac{1}{39}$
   (b) $\frac{1}{120}$
   (c) $\frac{1}{60}$
   (d) 10
   (e) $\frac{1}{12}$
   (f) 20
   (g) 3
   (h) $\frac{1}{21}$
   (i) 40
   (j) $\frac{8}{9}$
   (k) $\frac{4}{5}$
   (ℓ) $\frac{63}{160}$
   (m) $\frac{2}{9}$
   (n) $\frac{3}{10}$
   (o) $\frac{1}{3}$

3. See answers for 2.

4. (a) $\frac{64}{3}$ (d) $\frac{22}{3}$

(b) $\frac{80}{9}$ (e) $\frac{3}{2}$

(c) 70 (f) $\frac{9}{7}$

5. $\frac{3}{4}$

6. $\frac{1}{2}$

7. (a) $\frac{1}{3}$ cup orange juice

$\frac{1}{6}$ cup lemon juice

$\frac{1}{3}$ cup grapefruit juice

$\frac{2}{3}$ cup pineapple juice

1 cup water

$\frac{1}{3}$ cup syrup

(b) $\frac{2}{3}$ cup orange juice

$\frac{1}{3}$ cup lemon juice

$\frac{2}{3}$ cup grapefruit juice

$\frac{4}{3}$ cup pineapple juice

2 cups water

$\frac{2}{3}$ cup syrup

8. $16\frac{1}{4}$

6-7. Division of Rational Numbers

Care should be taken that the pupils understand that $\frac{\frac{2}{3}}{\frac{3}{4}}$ is just another way of writing $\frac{2}{3} \div \frac{3}{4}$. You may wish to point out that if the bar lengths are not carefully done, one may not be able to tell what is meant. Example $\begin{array}{c}2\\\hline 3\\\hline 4\end{array}$. Is this $\dfrac{\frac{2}{3}}{4}$ or $\dfrac{2}{\frac{3}{4}}$?

The procedure in this section is to show that our previous properties and definitions lead us from $\dfrac{\frac{3}{2}}{\frac{5}{7}}$ to $\frac{7}{5} \cdot \frac{3}{2}$, which generalizes to the usual rule: $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$. Again pupils should not be expected to reproduce this procedure but merely to see that we are led to the rule.

Answers to Exercises 6-7

1. (a) 1
   (b) $\frac{1}{9}$
   (c) 16
   (d) 72
   (e) $\frac{1}{100}$
   (f) 1
   (g) $\frac{25}{81}$
   (h) $\frac{1}{1000}$
   (i) $\frac{1}{2}$
   (j) $\frac{1}{12}$
   (k) $\frac{2}{3}$
   (ℓ) $\frac{3}{2}$

2. (a) $\frac{1}{10}$
   (b) $\frac{3}{4}$
   (c) $\frac{1}{72}$
   (d) $\frac{9}{4}$
   (e) $\frac{4}{3}$
   (f) $\frac{9}{4}$
   (g) $\frac{2}{17}$
   (h) 1
   (i) 64

3. (a) 2
   (b) 75
   (c) 8
   (d) $1\frac{1}{2}$
   (e) $2\frac{2}{3}$
   (f) $2\frac{1}{4}$
   (g) $7\frac{1}{2}$
   (h) $1\frac{3}{4}$
   (i) 6

4. (a) $\frac{1}{9}$
   (b) $\frac{3}{11}$
   (c) 1
   (d) $\frac{8}{9}$
   (e) $\frac{8}{9}$
   (f) $\frac{2}{3}$
   (g) $\frac{9}{10}$
   (h) 4
   (i) $\frac{2}{15}$

5. (a) $\frac{4}{3}$
   (b) $\frac{4}{7}$

6. (a) $\frac{7}{9}$
   (b) $\frac{3}{4}$

7. (a) 10 times as large
   (b) $\frac{16}{49}$ times as large

6-8. Addition and Subtraction of Rational Numbers

The purpose of this section is to strengthen the pupils understanding of these two operations by seeing justification of the methods of procedure with which they are already familiar.

Pupils can see that the methods they learned in 5th and 6th grade to add and subtract rational numbers are correct. They also see that there are many other methods of solution, some more applicable to one example than to another.

This section gives practice in the use of the least common multiple introduced in Chapter 5. It also illustrates how other names for the same number are extremely useful in the addition and subtraction of rational numbers.

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Encourage pupils to compute mentally as much as possible. This may be the pupils first introduction to a horizontal rather than vertical pattern in writing down work as a problem is solved. Encourage use of the horizontal form. The habit of using the vertical form may be strong. This habit may be difficult for some to overcome. Use of the horizontal form may encourage pupil to do more mental work.

Answers to Exercises 6-8a

1. (a) $\frac{6}{3}$ or 2
   (b) $\frac{6}{3}$ or 2
   (c) $4\frac{3}{8}$ or $\frac{35}{8}$
   (d) $4\frac{3}{8}$ or $\frac{35}{8}$
   (e) $\frac{7}{10}$
   (f) $\frac{7}{10}$
   (g) $\frac{17}{16}$ or $1\frac{1}{16}$
   (h) $\frac{17}{16}$ or $1\frac{1}{16}$

2. Yes.

3. (a) $\frac{7}{3}$ or $2\frac{1}{3}$
   (b) $\frac{7}{3}$ or $2\frac{1}{3}$
   (c) $\frac{15}{8}$ or $1\frac{7}{8}$
   (d) $\frac{15}{8}$ or $1\frac{7}{8}$
   (e) $3\frac{11}{30}$ or $\frac{101}{30}$
   (f) $\frac{101}{30}$ or $3\frac{11}{30}$
   (g) $\frac{52}{15}$ or $3\frac{7}{15}$
   (h) $\frac{52}{15}$ or $3\frac{7}{15}$

4. Yes.

*5. (a) Yes. (b) The commutative property.

*6. (a) Yes. (b) The associative property.

7. (a) $1\frac{5}{8}$ yards (b) 39¢

8. $1\frac{1}{8}$ pounds

Answers to Exercises 6-8b

1. (a) 4 (b) 100 (c) $53\frac{5}{6}$ (d) $37\frac{4}{5}$

2. (a) $7\frac{5}{8}$ (b) $10\frac{2}{16}$ or $10\frac{1}{8}$ (c) $17\frac{3}{32}$

3. (a) $\frac{2}{32}$ or $\frac{1}{16}$ (g) $\frac{7}{30}$
   (b) $1\frac{11}{16}$ (h) $\frac{7}{30}$
   (c) $\frac{19}{14}$ or $1\frac{5}{14}$ (i) $\frac{5}{12}$
   (d) $\frac{113}{64}$ or $1\frac{49}{64}$ (j) $\frac{5}{12}$
   (e) $\frac{129}{21}$ or $6\frac{1}{7}$ (k) 16
   (f) $\frac{231}{1000}$ (ℓ) $7\frac{1}{20}$

4. (a) They are equal. (b) They are equal.

5. (a) $\frac{7}{8}$ (b) 15 (c) $\frac{ad + bc}{bd}$

6. (a) $\frac{3}{5}$ (b) 15

7. No.

8. Sums of columns, rows, and diagonals are $22\frac{1}{2}$.

9. 5¢

Answers to Class Exercises

1. (a) $\frac{2}{4}$ or $\frac{1}{2}$ (g) $\frac{ad - bc}{bd}$
   (b) $\frac{1}{6}$ (h) $\frac{11}{35}$
   (c) $\frac{7}{8}$ (i) $\frac{13}{100}$
   (d) $\frac{9}{16}$ (j) $\frac{4}{32}$ or $\frac{1}{8}$
   (e) $\frac{5}{16}$ (k) $\frac{15}{18}$ or $\frac{5}{6}$
   (f) $\frac{3}{12}$ or $\frac{1}{4}$ (ℓ) $\frac{1}{1000}$

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(m) $\frac{1}{12}$

(n) $\frac{2}{8}$ or $\frac{1}{4}$

(o) $\frac{4}{9}$

2. (a) $1\frac{1}{8}$, 1.125, $\frac{18}{16}$, etc.

   (b) $\frac{11}{4}$, 2.75, $\frac{22}{8}$, $2\frac{6}{8}$, etc.

   (c) $1\frac{3}{5}$, 1.6, $\frac{16}{10}$, etc.

   (d) $\frac{17}{3}$, 5.666..., $\frac{34}{6}$, $5\frac{8}{12}$, etc.

Answers to Exercises 6-8c

1. (a) $\frac{8}{64}$ or $\frac{1}{8}$

   (b) $\frac{15}{16}$

   (c) $\frac{2}{9}$

   (d) $3\frac{7}{15}$ or $\frac{52}{15}$

   (e) $2\frac{15}{16}$ or $\frac{47}{16}$

   (f) $1\frac{1}{9}$ or $\frac{10}{9}$

   (g) $1\frac{1}{4}$ or $\frac{5}{4}$

   (h) $14\frac{1}{3}$ or $\frac{43}{3}$

   (i) $\frac{791}{1000}$ or 0.791

   (j) $\frac{4}{9}$

   (k) $\frac{4}{9}$

   (l) They are equal.

   (m) Yes.

2. Jane lives farther because $3 \cdot 16 > 5 \cdot 7$.

*3. A Mel's home
B Vic's home
C Bob's home
B is $\frac{3}{72}$ or $\frac{1}{24}$ of a mile farther than A.
C is $\frac{11}{72}$ of a mile farther than A.

4. $14\frac{3}{4}$ rods

5. $\frac{2}{5}$ second

6. $3\frac{1}{8}$ yard

7. $\frac{3}{4}$ hour or 45 minutes

8. (a) raised
   (b) $\frac{1}{6}$¢ per pound

9. (a) Less
   (b) If the difference were exactly 11, the larger number would be $6\frac{3}{4} + 11$ or $17\frac{3}{4}$. Since $17\frac{1}{4} < 17\frac{3}{4}$, the difference is less than 11.

10. In general, yes, unless the number subtracted is zero, or the original fraction has a value of 1.

11.

$\frac{2}{3}$	$\frac{1}{12}$	$\frac{1}{2}$
$\frac{1}{4}$	$\frac{5}{12}$	$\frac{\text{[illegible]}}{12}$
$\frac{1}{3}$	$\frac{3}{4}$	$\frac{1}{6}$

The sum in each row, each column and each diagonal is $\frac{5}{4}$.

6-9 and 6-10. Ratio and Decimals

These two sections should be considered as an introduction to two important uses of rational numbers. Both of these the students have met before. Here they are shown that a ratio is another way of thinking of a rational number and that a decimal is a way of writing a fraction having a power of 10 in its denominator. Decimals and fractions are not separate topics but should be treated together. (In fact, the old term was "decimal fractions".)

The purpose of the applications is to give the student a little idea of why these notions are useful. This is as far as this chapter is intended to go. Any mastery over either ratio

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or decimals will come later. If the teacher feels, for instance, that any problem having to do with rates would be too difficult for his class, he may well omit this topic here.

It is in Chapter 9 that ratio and decimals are taken up more thoroughly after the ideas of this chapter have been given an opportunity to "set" for awhile. They are also treated in the eighth grade in various appropriate places. These topics are much too important to be relegated to a single chapter and then forgotten. The full appreciation of their relationship with each other and later with percent will come in good time if the teacher continually works toward this end. At this stage many applications would only obscure the fundamental mathematical relationships. As the pupil acquires more experience in his science courses in school and his informal education outside the classroom, he builds a body of knowledge from which more applications can be drawn.

On the other hand, it is appropriate to introduce these topics here as preparation for what comes later and from the same point of view.

Ratio and decimals will be treated much more extensively in Chapter 9. It is not expected that students will acquire much skill with these topics in this brief introduction. It would be quite satisfactory not to include these topics in the chapter test. Rate as one of the applications of ratio is included to emphasize that ratio is a number and is not associated with a unit of measure. As ratio is used in a problem situation, it may be necessary to interpret an answer obtained in terms of a unit such as miles per hour or cents per article. The latter, for example, would arise in a problem about the cost of 15 pencils if six pencils sold for a quarter. One can think of the ratio of the number of cents to the number of pencils which is $\frac{25}{6}$. Rate, as an application, was chosen for this section, also because the formula, $d = rt$, is mathematically exactly the same sentence as $bx = a$, which is a basic consideration of the chapter. Similarly, $cn = t$, a formula for the cost of n

articles which sell for c cents per article, illustrates a use of the sentence $bx = a$.

Answers to Exercises 6-9

1. (a) $\frac{3}{10}$ (e) $\frac{5}{8}$

   (b) $\frac{5}{6}$ (f) $\frac{6}{5}$

   (c) $\frac{14}{3}$ (g) $\frac{75}{127}$

   (d) $\frac{9}{7}$ (h) $\frac{18}{11}$

2. (a) 520 miles per hour

   (b) $\frac{2600}{5} = \frac{520}{1} = 520$

3. There is a small advantage in using the same unit in numerator and denominator (which is not always possible). By so doing errors may be avoided. However, in a scale like 1 inch to 100 miles it would be quite inconvenient to change 100 miles to inches.

4. (a) $\frac{1}{20}$ (b) 85

5. 24" long, 16" wide

6. (a) $\frac{1}{300}$

   (b) 3600

   (c) $\frac{5}{2}$ in.

7. (a) $\frac{1}{6}$

   (b) 120 feet

   (c) $\frac{1}{6}$

8. 3

9.

10	12	16
25	30	40

10. (a) $\frac{3}{2}$ (b) $\frac{67}{40}$

11. 27, $\frac{99}{4}$

12. (a) $\frac{10}{11}$ (b) $60

*13. $\frac{22}{7}$

*14. (a) $\frac{67}{74}$
   (b) $\frac{74}{67}$
   (c) $\frac{67}{74}$

Answers to Exercises 6-10

1. (a) 0.5
   (b) 0.25
   (c) 0.375
   (d) 0.7
   (e) 0.34
   (f) 1.25
   (g) 2.75
   (h) 1.6
   (i) 0.625
   (j) 0.0625
   (k) 0.9375
   (ℓ) 3.7500

2. (a) $\frac{3}{4}$
   (b) $\frac{7}{4}$
   (c) $\frac{3}{5}$
   (d) $\frac{28}{5}$
   (e) $\frac{9}{25}$
   (f) $\frac{59}{25}$
   (g) $\frac{7}{50}$
   (h) $\frac{3}{250}$
   (i) $\frac{33}{40}$
   (j) $\frac{7}{8}$
   (k) $\frac{753}{500}$
   (ℓ) $\frac{251}{125}$

3. (a) Seventy-five hundredths
   (b) One and seventy-five hundredths
   (c) Six-tenths
   (d) Five and six-tenths
   (e) Thirty-six hundredths
   (f) Two and thirty-six hundredths
   (g) One hundred forty thousandths
   (h) Twelve thousandths
   (i) Eight hundred twenty-five thousandths
   (j) Eight hundred seventy-five thousandths
   (k) One and five hundred six thousandths
   (ℓ) Two and eight thousandths

4. $50.53

5. $\frac{42}{8}$

1.25, 0.5, 0.375, 2.5, 0.625

5.25

6. 28.996

7. (a) 32.5
   (b) 8.1

8. (a) 0.04
   (b) 2.5
   (c) 0.356
   (d) 2.40
   (e) 14.92
   (f) 67.3
   (g) 0.001
   (h) 2.76

*9. (a) 0.667
   (b) 0.833
   (c) 0.444
   (d) 0.091
   (e) 0.111
   (f) 2.375
   (g) 13.667
   (h) 7.857
   (i) 166.667
   *(j) 101.620

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*10. $3.25 = 325(0.01) \qquad 6.71 = 671(0.01)$

$$3.25 + 6.71 = 325(0.01) + 671(0.01)$$

$$= (325 + 671)(0.01) = 996(0.01) = 9.96$$

6-11. Ordering

The chief object of this section is to give a means of comparing two fractions. The method of finding the decimal equivalents of the two fractions is probably a little longer to write out than the other method but it also may be easier to understand. The pupils should understand both and then be allowed to choose that which they prefer.

It is not thought necessary to give detailed explanation of why it is true that when two fractions have equal denominators, that with the greater numerator represents the greater number. If some students have trouble with this it might be explained, for example, that

$$\frac{3}{10} > \frac{2}{10}$$

because

$$3 \times \left(\frac{1}{10}\right) > 2 \times \left(\frac{1}{10}\right)$$

Probably the simplest method of comparing fractions is not mentioned. If in Section 5 of this chapter the pupils found that $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$, then it should not be hard for them to deduce and use

$$\frac{a}{b} > \frac{c}{d} \quad \text{if and only if} \quad ad > bc.$$

This is probably the simplest test of all, but it is not mentioned because if not understood it often becomes purely mechanical.

Answers to Exercises 6-11

1. (a) $\frac{2}{3} = \frac{8}{12}$ and hence $\frac{2}{3} > \frac{7}{12}$.

   (b) $\frac{4}{5} = \frac{64}{80}$ and $\frac{13}{16} = \frac{65}{80}$. Hence $\frac{13}{16} > \frac{4}{5}$.

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(c) $\frac{13}{5} = \frac{91}{35}$ and $\frac{13}{7} = \frac{65}{35}$. Hence $\frac{13}{5} > \frac{13}{7}$.

The bright student might find the answer more quickly by noticing that the numerators are equal and hence the larger fraction is the one with the smaller denominator, since the denominator denotes the number of equal parts into which the numerator is divided. See Problem 5.

2. One must compare $\frac{5}{11}$ with $\frac{4}{9}$. Now $\frac{5}{11} = \frac{45}{99}$ and $\frac{4}{9} = \frac{44}{99}$ and hence $\frac{5}{11} > \frac{4}{9}$.

3. For brand A, one gets $\frac{15}{23}$ of an ounce for one cent; for brand B, $\frac{8}{12}$ of an ounce, that is, $\frac{2}{3}$ of an ounce. Now $\frac{2}{3} = \frac{46}{69}$ and $\frac{15}{23} = \frac{45}{69}$. Hence $\frac{2}{3} > \frac{15}{23}$ and brand B is cheaper. The same result may be found by comparing the the costs per ounce, that is, $\frac{23}{15}$ and $\frac{12}{8}$.

4. The pupil might answer this question by giving several examples showing the following which, for the benefit of the teacher we show in letters: $a > 2b$ implies that $\frac{a}{b} > \frac{2b}{b} = 2$.

*5. If two fractions have equal numerators, the fraction with the smaller denominator represents the larger number. A satisfactory reason from the pupil's point of view at this stage should be that the denominator denotes the number of equal parts into which the numerator is divided; if the number of parts is smaller, each part must be larger. It is too early to discuss manipulations with inequalities.

Sample Questions for Chapter 6

I. True-False Questions

(T) 1. The product $\frac{3}{7} \cdot \frac{3}{3}$ is equal to $\frac{3}{7}$.

(F) 2. Whole numbers are not rational numbers.

(T) 3. $\frac{8\frac{1}{3}}{100} = \frac{1}{12}$.

(T) 4. In adding rational numbers, if the denominators of the fractions are equal we add the numerators.

(T) 5. The following numbers are all examples of rational numbers:

$$\frac{3}{4},\ 5,\ \frac{8}{3} \text{ and } 1\frac{1}{2}.$$

(T) 6. Zero is the identity element for addition of rational numbers.

(T) 7. The fractions $\frac{0}{a}$ and $\frac{0}{b}$ represent the same rational number if neither a nor b is zero.

(F) 8. If a and b are rational numbers, $\frac{a}{b}$ is always a rational number. (Note: 0 is a rational number; except for 0 the statement is true.)

(T) 9. A rational number multiplied by its reciprocal equals 1. (Note: If it is zero, it has no reciprocal.)

(T) 10. The symbol $\frac{24}{8}$ stands for a number which is both a whole number and a rational number.

(F) 11. The sum of two rational numbers whose fractions have equal numerators may be found by adding their denominators.

(T) 12. The product of zero and any rational number is zero.

(F) 13. If one fraction has a larger numerator than a second fraction, the number represented by the first fraction is always larger than the number represented by the second fraction.

(T) [illegible]4. Even if $a = 0$, $\frac{a}{7}$ is a rational number.

(F) 15. If two fractions have the same denominator, the numbers they represent are always equal.

(F) 16. The reciprocal of $\frac{13}{19}$ is $\frac{1}{\frac{19}{13}}$.

(F) 17. The least common multiple of the denominators of $\frac{1}{2}$ and $\frac{5}{6}$ is 12.

(T) 18. In the division problem $\frac{1}{2}$ divided by $\frac{1}{3}$, we are looking for a number which when multiplied by $\frac{1}{3}$ gives $\frac{1}{2}$.

(F) 19. In the division problem $\frac{1}{2}$ divided by $\frac{1}{3}$, we are seeking a number which when multiplied by $\frac{1}{2}$ gives $\frac{1}{3}$.

(F) 20. The reciprocal of the reciprocal of 3 is $\frac{1}{3}$.

(F) 21. Even if b equals 0, $\frac{a}{b}$ is a rational number.

(F) 22. The sum: $\frac{a}{c} + \frac{b}{c}$ is equal to $\frac{a + b}{2c}$.

II. Multiple Choice.

23. The sum: $\frac{r}{s} + \frac{t}{u}$ is equal to which of the following for all counting numbers r, s, t and u:

a. $\frac{r + t}{s + u}$

b. $\frac{r + t}{su}$

c. $\frac{rs + tu}{su}$

d. $\frac{st + ru}{su}$

e. None of these

23. D

24. Which of the following pairs of numbers are both divisible by the same number greater than one?

a. 7, 3.
b. 8, 9.
c. 7, 28.
d. 5, 23.
e. None of these.

24. C

25. The product: $\frac{x}{z} \cdot \frac{t}{k}$ is equal to which of the following if x, t, z and k are counting numbers:

a. x plus z plus t plus k.
b. $\frac{xk}{zt}$
c. $\frac{xt}{zk}$
d. $(xt)(zk)$
e. None of these

25. C

26. If $\frac{x}{a} = \frac{y}{b}$ and $a = 6$ and $b = 12$, then

a. $x = 2y$
b. $y = 2x$
c. $6x = y$
d. $12x = 12y$
e. None of these

26. B

27. We can change the denominator of the fraction $\frac{\frac{2}{3}}{\frac{4}{5}}$ to the number "1" without changing the number represented by the fraction by:

a. Adding $\frac{5}{4}$ to the numerator and denominator.
b. Subtracting $\frac{5}{4}$ from the numerator and denominator.
c. Multiplying both the numerator and denominator by $\frac{5}{4}$.
d. Dividing both the numerator and denominator by $\frac{5}{4}$.
e. None of these.

27. C

28. Given the five numbers: $0.12,\ \frac{1}{4},\ \frac{1}{8},\ 0.099,\ \frac{2}{11}$.

The smallest is:

a. 0.12 d. 0.099
b. $\frac{1}{4}$ e. $\frac{2}{11}$
c. $\frac{1}{8}$

28. D

29. Given the same five numbers as in Question 28. The largest is which of those indicated?

29. B

III. General Questions. In most cases the student should give reasons. In each case below insert one of <, =, > so as to make the statement true:

			Answers
30.	$\frac{7}{8}$	$\frac{5}{8}$	>
31.	$\frac{3}{8}$	$\frac{3}{9}$	>
32.	$\frac{3}{2}$	$\frac{2}{3}$	>
33.	$\frac{9}{9}$	$\frac{7}{7}$	=
34.	$\frac{6}{20}$	$\frac{11}{35}$	<
35.	$\frac{0}{5}$	$\frac{0}{3}$	=
36.	$\frac{19}{20}$	$\frac{18}{19}$	>

37. Express as a single decimal:

		Answers
a.	$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000}$	0.111
b.	$\frac{1}{10} + \frac{10}{100} + \frac{100}{1000}$	0.3
c.	$\frac{1}{2} + \frac{1}{5} + \frac{1}{4}$	0.95
d.	$\frac{1}{10} + \frac{1}{4} + \frac{1}{5}$	0.55

38. Write each of the following in decimal form:

		Answers
a.	$\frac{13}{100}$	0.13
b.	$\frac{13}{1000}$	0.013
c.	$\frac{13}{10}$	1.3
d.	$\frac{130}{10000}$	0.013

39. Write each of the following as a fraction with counting numbers in numerator and denominator.

		Answers
a.	0.15	$\frac{15}{100} = \frac{3}{20}$
b.	0.013	$\frac{13}{1000}$
c.	1.25	$\frac{125}{100} = \frac{5}{4}$
d.	15.12	$\frac{1512}{100} = \frac{378}{25}$

40. Show by changing decimals into fractions:

a. $15.1 \cdot 100 = 1510$

b. $\frac{15.1}{1000} = 0.0151$

c. $12.3 \times 0.5 = 6.15$

d. $\frac{1.5}{0.5} = 3$

41. There are 40 questions on a test. If all questions are given the same value and if a perfect paper gets a grade of 100, how much should each question count? How many questions would a student have to answer correctly to get a grade of 90 or better? Ans: $2\frac{1}{2}$, 36

42. When a merchant buys candy bars, he pays 40 cents for boxes holding 25 bars. If he sells them at 2 bars for 5 cents, what is his profit on each bar?
Ans: Each bar costs 1.6 cents, he sells them for $2\frac{1}{2}$ cents each. Hence his profit is 0.9 cents each. This can also be found using fractions exclusively.

43. A crafts class needs a type of decoration that sells for $7\frac{1}{2}$ cents a foot in one shop and at 3 feet for 25 cents in another shop. How much can be saved on each foot at the cheaper price?
Ans: At the second shop the price is $8\frac{1}{3}$ cents for a foot and hence $\frac{5}{6}$ of a cent per foot can be saved by buying at the first shop.

44. A group of seventh graders have promised to collect 50 pounds of scrap metal. They have $36\frac{7}{8}$ pounds; how much more must they collect to keep their promise?
Ans: $13\frac{1}{8}$.

45. Tom needs four pieces of wood $2\frac{3}{4}$ feet long for the legs of a table. Boards from which this wood can be cut come in the following lengths: 8 feet, 10 feet, 12 feet. What length board should he get and how much will be left over?
Ans: He needs 11 feet; hence he should get the 12 foot board and will have one foot left over.

46. A boy's car used 15 gallons of gasoline for a 225 mile trip. How many miles did he drive for each gallon of gasoline used? Ans: 15

47. If 6 inches on a map represents 90 miles, how many miles does each inch represent? Ans: 15 miles

Find the value of x in simplest form which makes the following true:

		Answers
48.	$x = 0 \cdot (\frac{1}{3} + \frac{1}{2} + \frac{1}{6})$	$x = 0$
49.	$x = \frac{1}{3} \div (\frac{1}{4} + \frac{1}{5})$	$x = \frac{20}{27}$
50.	$\frac{x}{3} = \frac{6}{9}$	$x = 2$
51.	$x = \frac{91 - 35}{7}$	$x = 8$
52.	$x =$ product of $\frac{7}{10}$ and $\frac{10}{10}$	$x = \frac{7}{10}$

		Answers
53.	$x = \frac{165}{147}$	$x = \frac{55}{49}$
54.	$\frac{x}{16} = \frac{7}{4}$	$x = 28$
55.	$x = (3\ \cdot 5)\ \cdot (\frac{1}{5 \cdot 3})$	$x = 1$
56.	$x = \frac{11}{24} \div \frac{11}{2}$	$x = \frac{1}{12}$
57.	$x = \frac{2}{3} - \frac{3}{5}$	$x = \frac{1}{15}$

CHAPTER 7

MEASUREMENT

Introduction

The idea of measurement is of fundamental importance in modern life. This is true in the day by day activities of ordinary citizens and equally true in the activities associated with most vocations. Newspapers and other kinds of reading matter are full of references to measurements of one kind or another. Consequently, there is a great deal of information about the topic which could conceivably be taught in the seventh grade, and which some, if not all, seventh grade pupils would find both useful and interesting. The ability to understand what other people are doing depends to a considerable extent on understanding the kinds of things they measure and the techniques of measurement which are used. This in turn depends upon an understanding of the nature of the thing measured. Adequate treatment of many techniques of measurement is therefore more suitable content for courses in other fields, such as science, than in mathematics.

The fundamental notion of measurement, however, is common to measurement in all fields, and the development of basic concepts is therefore an important topic in the mathematics curriculum. The measurement of one-, two-, and three-dimensional figures is also essential both for development of certain geometric concepts and for applying these concepts. Furthermore, many things which are not geometric in character are measured by relating their properties to linear or circular scales. In this chapter, therefore, attention is focused chiefly upon development of basic concepts which underlie measurements of all kinds, and upon measurements of line segments, surfaces, solids, and angles.

While pupils entering the seventh grade have had a substantial amount of experience with measurement, some of them may have no clear understanding of the basic concepts and assumptions which underlie the process. Among the concepts developed in this chapter are the following:

1. The size of a collection of discrete objects is found by counting and described by numbers; the size of a continuous quantity is found by measuring and described by a measurement which contains both a number and a unit of measurement.

2. The process of counting separate objects yields a whole number which represents exactly the size of the collection; the process of measuring yields a number of units which is never exact, but is an *approximate* representation of the size of the quantity measured.

3. Since all measurements are approximate, a measurement should be reported so as to indicate its *precision*, or its *greatest possible error*. The precision is shown by naming the smallest unit or subdivision used. This implies the size of the greatest possible error, or the interval within which the true measurement lies. Pupils should be encouraged to report measurements which they have made in a way consistent with this idea, but cannot be expected at this stage to understand fully its implications for computed measurements.

4. As suggested above, development of a method for measuring anything rests upon understanding the aspect of it which is to be measured. This is true of the measurement of geometric continuous quantities to which a large part of this chapter refers. Therefore, four assumptions regarding the nature of geometric continuous quantities are stated.

5. A unit of measurement for geometric quantities must be of the same kind as the thing to be measured--a unit line segment to measure a line segment, a unit surface to measure a surface.

6. The size of a unit of measurement is arbitrary.

7. The measure of a geometric quantity is obtained by subdividing it into parts the size of the unit, and counting the number of unit parts.

8. Standard units of measurement are necessary in highly organized societies, and are established by social agreement.

9. It is convenient to have related units for length, surface, and volume, such as the inch, square inch, and cubic inch.

10. The area of a region enclosed by a simple closed curve, such as the rectangle, may be found by computation from lengths rather than direct measurement of surface; the same is true for the volume of a rectangular solid.

11. Just as there are many names for the same number, there are many names for the same measurement.

12. Many quantities of different nature from those stressed in this chapter can be measured when suitable units are used.

7-1. Counting and Measuring

Section One develops the difference between discrete objects and continuous quantities. Four properties of geometric figures are developed at this time:

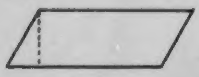
1. The motion property states that figures can be moved without changing size or shape.
2. The comparison property states that the sizes of two geometric quantities can be compared provided these quantities have the same nature.
3. The matching property states that two geometric quantities have the same size if every part of one can be matched to a part of the second so that no part of either figure is omitted.
4. Finally, it is stated that a geometric figure may be subdivided.

These four properties are the basis of measurement. The teacher should be alert in later sections to point out that the measuring process utilizes these properties. Measuring is done in this section but many students will not recognize it as such since standard units are not used.

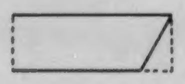
Problems 1 and 2 develop the concept of continuous quantities in contrast to discrete objects. Problems 3 to 5 develop the concept of measuring as a comparison to a unit having the same nature as the object measured. Problems 6 and 7 direct attention to the possibility that different sized closed regions may be bounded by closed curves of the same length, and that a closed region of a given size may be enclosed by curves of different lengths.

Answers to Exercises 7-1

1. (a) Counting
   (b) Measuring
   (c) Counting
   (d) Measuring
   (e) Measuring
   (f) Measuring
2. (a) Continuous
   (b) Continuous
   (c) Discrete
   (d) Continuous
   (e) Discrete
   (f) Continuous
3. (a) Congruent
   (b) $\overline{RT}$
   (c) Congruent
   (d) $\overline{RT}$
   (e) $\overline{TP}$
4. Closed region C < Closed region A
   Closed region C < Closed region B
   Closed region C < Closed region D
5. (a) All are approximately equal.
   (b) To compare A with B, cut along shorter diagonal of A.
   To compare A with C, turn A over.
   To compare A with D, cut as follows:



   and reassemble as follows:



   (c) All should have the same area.
6. (a) Curves have the same length.
   (b) Closed region E is larger than closed region F.

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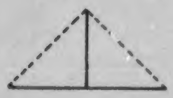
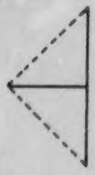
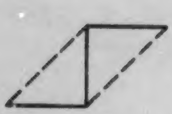
7. (a) Pupils will probably assemble the parts like this:

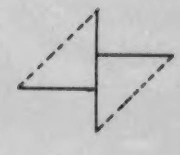


or



or





Equally correct: or any other rearrangement. Areas of new figure and area of the square are the same.

(b) Lengths of the curves will vary but all will be longer than the original square. The diagonal is longer than a side. Figures like the last one increases the length of the curve by an additional length.

7-2. Subdivision and Measurement

Measuring geometric continuous quantities is accomplished by the process of subdividing the quantities into unit parts. We think of geometric figures as keeping both size and shape when they are moved. This is important for comparison of two geometric quantities, and also for subdividing a quantity into parts.

There is the hidden assumption in the Comparison Property that, for two geometric quantities a and b, $a = b$, a is greater than b, or a is less than b, and that exactly one of these relations is true. By using the Motion Property it is possible to determine which of these relations is true for any two quantities of the same kind.

The Matching Property is basic to the notion that differently shaped polygons, or other simple closed curves, may have equal perimeters or form closed regions having the same area. It leads to the Subdivision Property which states the assumption that a quantity may be subdivided without producing a change in the size of the whole. All of these properties are used in the process of measuring.

Note that we do not define such concepts as length, area, volume, etc. We attempt to lead the students toward an intuitive grasp of these concepts.

The paragraphs on Subdividing Continuous Quantities illustrate with line segments the essential notion that a unit segment "n" may be laid off repeatedly on a given segment to locate a point such that the segment remaining is less than one unit. The concepts developed in Sections 1 and 2 provide the basis for the definition of the terms "measure" and "unit of measurement." Note that, according to the definition, the measure is a number, while the length is defined as a phrase which includes both the measure and the unit.

Actually we take a measurement of a length. The more precise our measurement is the closer we come to the length. However, the measurement is not the exact length since we cannot possibly hope to achieve absolute precision. Hence, when we say the length is 4n we mean the measurement is 4n. The length is $\approx$ 4n. Here we have a clear-cut example of the distinction between the abstract mathematical notion of length and the fitting of this notion to the physical world through measurement. This is why we say all measurements are approximate.

Answers to Exercises 7-2

1. Length of $\overline{AB} \approx$ 4c
2. (a) Size of closed region A $\approx$ 3 times size of closed region B.
   (b) 3
3. (a) Length of curve D $\approx$ 2 times the length of curve C.
   (b) Closed region D $\approx$ 3 times closed region C.
4. 3 Note: The definition of rectangular solid appears in Section ~~2~~ of Chapter 8 where a more formal discussion

of volume is developed.

5. (a) Segment of length c.
   (b) 4

6. (a) Closed region B.
   (b) 3

7. (a) 6t
   (b) 12t
   (c) Length of curve D is 2 times the length of curve C.
   (d) Closed region C is the unit; closed region D is 3 times closed region C.

8. Unit of measure was rectangular solid B; rectangular solid A is 3 times rectangular solid B.

9. (a) DE, AF, FC are approximately the same length.
   AD, DB, EF are approximately the same length.
   BE, EC, DF are approximately the same length.
   (b) ADF, FEC, DBE, DEF, ADEF, FDEC, DBEF, ADEC, DFCB, AFEB, ABC.
   (c) $\triangle$ ADF, $\triangle$ DEF, $\triangle$ DBE all have the same area as $\triangle$ EFC. The closed triangular region ABC is 4 times the closed triangular region EFC.
   (d) Closed triangular regions ADF, EFC, DEF, DBE have the same area.
   Closed quadrilateral regions ADEF, FDEC, DFEB have the same area.
   Closed quadrilateral regions ADEC, FDBC, and EFAB have the same area.

7-3. Subdividing Units of Measurement

This section strengthens the basic concepts of measuring and provides for utilizing rational numbers to express measures that do not appear to be multiples of the unit. The measured object has been subdivided previously. Now the measuring unit is subdivided in order to cover the whole surface.

The arbitrary nature of the measuring unit is emphasized as is also the idea that the measuring unit must have the same nature as the object measured.

The use of plane figures of many shapes as a unit should help the student appreciate the advantages of a square unit, and later of a cubical unit as the unit of measurement for three-dimensional figures. These figures are also used to point out that the measure is independent of the way the units are matched to the object.

Answers to Class Exercises 7-3

1. Answers will vary but should be reasonable.
2. Answers will vary but should be reasonable.
3. Differences arise from differences in unit of measurement, i.e., length of foot.
4. Book, chalk box, ream of paper, etc.
5. The measure is the number of sheets used but the size is the number and the name, "note-book sheets."
6. Measures varied.

Answers to Exercises 7-3

1\.

(a) The answer depends upon the size of sheet used. For an $8\frac{1}{2} \times 11$ sheet of paper, the computed numbers of units required are as follows:
Rectangular unit -- about 47
Isosceles right triangular unit -- about 47
Equilateral triangular unit -- about 54
Circular unit -- about 30

(b) The circular unit is probably hardest to use because of the space between the units.

(c) Same for rectangular units (2" × 1") and isosceles right triangular units with 2" legs.

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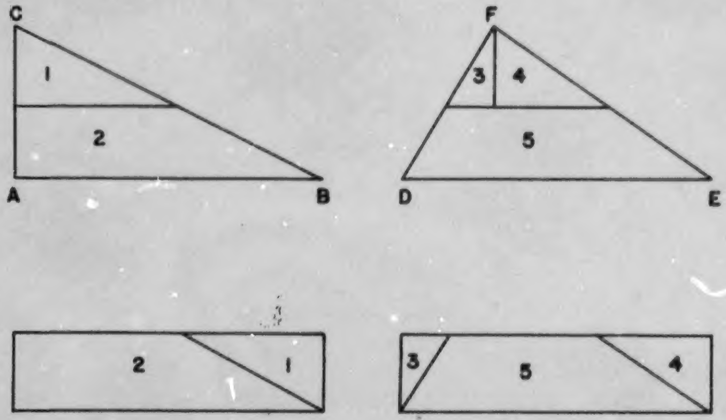
2. The measurements will differ, since the pupils' feet will differ in length.

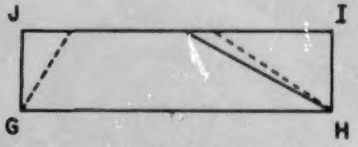
3. Marbles are a poor unit because of space between them. Rectangular blocks would be much better.

4. BRAINBUSTER.

(a)



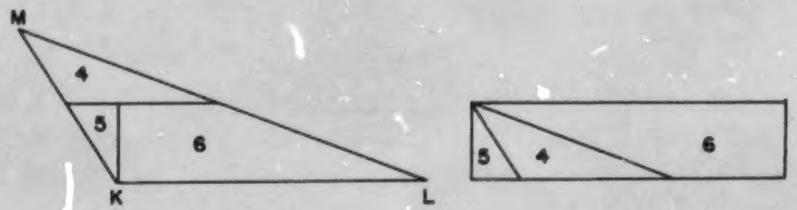
(b)



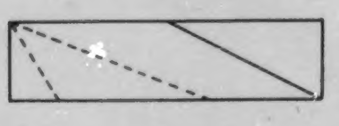
The solid line shows the arrangement of the parts of triangle ABC, and the dotted lines the arrangement of the parts of triangle DEF.

Note that the measure of $\overline{JG}$ is one-half the measure of the altitude of each triangle.

(c)



(d)



The solid line shows the parts of triangle ABC, and the dashed line the parts of triangle MKL.

7-4. Standard Units

The pupils have had many exercises to demonstrate the meaning of measurement and the arbitrary nature of the unit. This section should establish the social need for units that are the same for the entire group. Mass production and the convenience of interchangeable parts provide a wealth of material for pointing out this need. The historical material should be of interest and show society's increasing need for standard units.

Archimedes used density in determining the purity of gold. By experiment, he found that the same weights of silver and of gold displaced different volumes of water. By immersing gold that had the same weight as the crown and comparing the water displacement with that displaced by the crown, Archimedes showed that the crown was not all gold.

The metric system is briefly introduced in this chapter only because of the chapter's length; it will be discussed more fully in Chapter 3 of Volume II. It should be stressed that our linear units are defined, by law, in terms of the metric units. The metric system is legal in the U.S. If class time permits, a discussion of the new definition of a meter as 1,650,763.73 times the wave length of orange light from krypton 86 might be interesting.

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This standard for the meter is difficult to visualize, but has the advantage that it can be reproduced in any good scientific laboratory and is more precise than the platinum and iridium bar in France that is the present standard for the meter. This definition has been adopted by the Advisory Committee on the Definition of the Meter. Current Science for the week of September 29 - October 3, 1958 has an article on this at the student's level.

The approximate nature of measurement has been pointed out throughout the chapter. Continuing emphasis on the fact that measurements are not exact should be made, although this topic will be treated more formally in Section 5. Comparing measurements of the same object made by different students and also measurements determined by instruments marked with varying degrees of precision help develop the concept of the approximate nature of measurement.

Some 7th grade students have difficulty measuring with a ruler. Through making cardboard rulers and studying the increasingly fine divisions in Figure 7-4, it is hoped that the poor students will overcome their difficulties while the better student is seeing the relationship with base 2. The "Class Problems, 7-4" should be done orally in class. Some of the actual measuring should also be done in class so that the teacher can identify and work with students who have not mastered the use of the ruler.

In the hodge-podge that is the English system of measures, there is a variety of standard units. Conversions from one unit to another cause a great deal of trouble both in mathematics and in science. Conversions are treated as number sentences with a stress on the relation between the units involved in the conversion. This should be developed in class and class practice should be provided to make sure the students understand what they are doing. The use of estimates and common sense are both important aids in converting units. Measurements recorded in different units but treated as the same unit are the basis for many student errors. Pupils should be taught to read the names of units as well as the number of these units. Common sense is called on again to decide which would be the best unit to use for a particular problem.

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Exercises 7-4a

These exercises are designed to show the value of standard units to social living. Some of the questions are answered best by personal interview with local merchants. All values to the students could be realized by having each student do one or two of the exercises.

1. Most bread is bought by the loaf. Loaves are baked in various sizes and many bakeries indicate the weight on the wrapper.
2. The size of food cans is related to volume. Since a packer could leave space at the top of a can, the law requires that each can have the weight of the contents printed on it. Students might check different brands of the same item to see if the contents of the same size can always has the same weight.
3. A few items to suggest, if the students fail to find others, are: bicycles - diameter of the wheel; hats - diameter of head for men and perimeter for women; pans - capacity; screws - length and threads per inch; fishing rods - length; drill bits - diameter.

Exercises 7-4b

These exercises give the student practice in measuring and they develop the need for fine subdivision. It is a good preparatory exercise for those who have difficulty with reading a ruler.

1. and 2. Cardboard rulers can be checked with commercial rulers.

3. (a) $2\frac{1}{2}$" (b) $3\frac{1}{2}$" (c) 2" (d) 4" (e) 3"

4. (a) $2\frac{1}{2}$ (b) $3\frac{1}{4}$ or $3\frac{1}{2}$ (c) $2\frac{1}{4}$ (d) $3\frac{3}{4}$ (e) 3 or $3\frac{1}{4}$

5. (a) $2\frac{1}{2}$ (b) $3\frac{3}{8}$ (c) $2\frac{1}{8}$ or $2\frac{1}{4}$ (d) $3\frac{3}{4}$ or $3\frac{7}{8}$ (e) $3\frac{1}{8}$

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6. Answers will vary but should approximate those above.
7. 8^{ths} of an inch since the divisions come closer to matching the line segments.

Class Exercises 7-4

1. The need to measure segments smaller than an inch.
2. Into 2 parts.
3. Each section of the inch on the left of any given inch (except the 6th inch) is divided into two parts.
4. Each section is divided into two parts.
5. 4; 8; 16. (The third inch lies between 2 and 3.)
6. The line segments are longer.
7. Longer line segments help group the shorter line segments so that the number and size of the spaces are easier to see.
8. Theoretically, there is no limit to the number of divisions.

Exercises 7-4c

1. 32
2. (A) $\frac{3}{4}''$
   (B) $1\frac{5}{8}''$
   (C) $2\frac{7}{8}''$
   (D) $3\frac{3}{8}''$ or $3\frac{1}{2}''$
   (E) $4\frac{3}{8}''$
   (F) $5''$
   (G) $5\frac{5}{8}''$

3. (a) $\frac{7}{8}''$

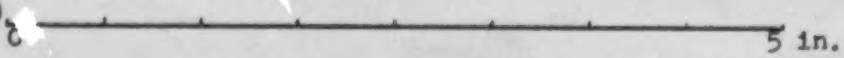
(b) $2\frac{1}{8}$

(c) $\frac{1}{2}''$ or $\frac{5}{8}''$

(d) $1\frac{1}{2}$ or $1\frac{5}{8}$

(e) $\frac{5}{8}''$

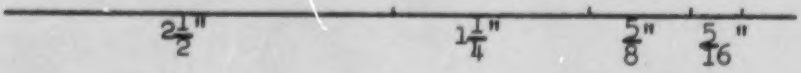
4. (a)



(b) 8

(c) The divisions on the line picture one meaning of $5 \div \frac{5}{8}$; there are 8 pieces.

5. (a)



(b) $4\frac{11}{16}$

(c) $4\frac{11}{16}$

6. (a) $\frac{7}{16}''$

(b) $1\frac{9}{16}$

(c) $5\frac{1}{2}$

(d) $2\frac{1}{2}''$

(e) $\frac{3}{4}''$

(f) $3\frac{1}{2}''$

(g) $\frac{1}{2}''$

(h) $\frac{5}{8}''$

7. (a) AB $\approx \frac{7}{16}''$

BC $\approx \frac{13}{16}''$

CD $\approx \frac{3}{16}$

DE $\approx \frac{9}{16}''$

EF $\approx \frac{3}{4}''$

FG $\approx 1''$

GH $\approx \frac{1}{2}''$

HI $\approx \frac{5}{8}''$

IJ $\approx \frac{5}{8}''$

(b) There is a possibility of a slight ($\frac{1}{16}''$) discrepancy due to the approximate nature of measures.

8.

Base 10		Base 2
$\frac{1}{2}$	=	$\frac{1}{10}$
$\frac{1}{4}$	=	$\frac{1}{100}$
$\frac{1}{8}$	=	$\frac{1}{1000}$
$\frac{1}{16}$	=	$\frac{1}{10,000}$

9. Since all divisions are obtained by dividing existing divisions by 2, the number of sections increases by powers of 2, or, in base 2 notation, by annexing a zero to show the number of divisions.

Answers to Exercises 7-4d

1. Answers will vary.

2. (a) $2\frac{1}{2}$" (d) $3\frac{13}{16}$"

   (b) $3\frac{3}{8}$" (e) $3\frac{1}{8}$

   (c) $2\frac{3}{16}$"

3. (a) $7\frac{1}{2}$" (d) $7\frac{3}{4}$"

   (b) 7" (e) $6\frac{13}{16}$

   (c) 9" (f) $7\frac{3}{4}$

4. (a) No.

   (b) Make the string fit the figure and then measure the string.

   (c) (1) $9\frac{7}{16}$" (computed) (3) $\approx$ 8"

   (2) $\approx$ 9" (4) $\approx$ 6"

5. (a) Length ≈ 5400 in. (Answers in feet are acceptable. See part b.)
Width ≈ 900 in.
Height ≈ 540 in.
(b) Length ≈ 450 ft. ≈ 150 yd.
Width ≈ 75 ft. ≈ 25 yd.
Height ≈ 45 ft. ≈ 15 yd.

*6. The S. S. United States is 990 ft. long. The dimensions given for the Ark make it surprisingly large for the engineering of that time. The length is greater than the length of a football field.

Answers to Class Exercises 7-4e

2. (a) ≈ 6cm. (d) ≈ 10cm.
(b) ≈ 9cm.
(c) ≈ 6cm.

3. (1) 2(6cm. + 3cm.) ≈ 18cm.
(2) 3cm. + 3cm. + 5cm. + 3cm. + 4cm. ≈ 18cm.
(3) 2(4cm.) + 4(3cm.) ≈ 20 cm.
(4) 8cm. + 5cm. + 4cm. + 3cm. ≈ 20 cm.

Answers to Class Exercises 7-4f

1. (a) ≈ 64mm. (d) ≈ 97mm.
(b) ≈ 86mm. (e) ≈ 80mm.
(c) ≈ 56mm.

2. (a) $2(58\text{mm} + 32\text{mm}) \approx 180\text{mm}$.
   (b) $33\text{mm} + 26\text{mm} + 50\text{mm} + 26\text{mm} + 38\text{mm} \approx 173\text{mm}$.
   (c) $2(65\text{mm} + 50\text{mm}) \approx 230\text{mm}$.
   (d) $50\text{mm} + 65\text{mm} + 83\text{mm} \approx 198\text{mm}$.
   (e) $2(39\text{mm}) + 4(25\text{mm}) \approx 178\text{mm}$.
   (f) $80\text{mm} + 50\text{mm} + 42\text{mm} + 26\text{mm} \approx 198\text{mm}$.
3. (b) $1 \text{ in.} \approx 2\frac{1}{2}\text{cm}$.
   (c) $1 \text{ in.} \approx 25\text{mm}$

7-5. Precision of Measurement and the Greatest Possible Error

This section is the first attempt in this book to formalize the approximate nature of measurement. Precision of measurement is introduced through work with the ruler. Precision, as such, is not defined. We do speak of one measurement as being more precise than another. At first, increasing precision of measurement is indicated by increasing denominators (of the fractions representing the subdivision being used). This was done so that the two "things" under discussion would both be "moving" in the same direction, i.e. increasing. Later the idea is introduced that increasing precision of measurement is associated with decreasing unit size (this counter motion can cause difficulty for the students).

Suppose we are given units A and B with B smaller than A. A measurement made with unit B is more precise than one made with A because the measurement obtained with unit B is closer to the true measurement than that obtained with unit A.

If an attempt is made to define precision as a noun we get into a situation where increasing precision is associated with decreasing accuracy. This should be avoided as it contradicts normal associations of these words.

Two notations are used for indicating precision of measurement.

1. Indicated by the form of the numeral and the name of the unit, e.g. $3\frac{2}{8}$ in. indicates measurement to the nearest $\frac{1}{8}$ in. This fraction must not be simplified to $3\frac{1}{4}$. This idea can be used later with decimals, e.g. 4.13 in. indicates measurement to the nearest $\frac{1}{100}$ in.
2. Indicated by using the greatest possible error and the name of the unit. This idea is used later in the discussion of precision of area measurement. The first notation cannot be used for area; this is pointed out in Section 7.

Answers to Class Discussion Exercises 7-5a

1. $3\frac{1}{2}$"
2. $3\frac{1}{4}$" or $3\frac{1}{2}$"
3. $3\frac{3}{8}$"
4. $3\frac{6}{16}$"
5. There is at least one point at which the measure may stay the same. A higher degree of precision may produce further change, if such rulers are available.
6. Line (c) 2"; $2\frac{1}{4}$"; $2\frac{1}{8}$" or $2\frac{1}{4}$"; $2\frac{3}{16}$"; same as Prob. 5.
7. Closer to $3\frac{3}{8}$" than to either $3\frac{2}{8}$" or $3\frac{4}{8}$". Nearest $\frac{1}{8}$".

Answers to Class Discussion Exercises 7-5b

1. (a) $\frac{1}{8}$; (b) yes;
   (c) No, this would indicate measurement to the nearest $\frac{1}{16}$ in.;
   (d) Precise to the nearest $\frac{1}{8}$ in.
2. Precise to the nearest $\frac{1}{16}$ in.

3. (a) $\frac{1}{16}$;

   (b) No, since precision to the nearest $\frac{1}{16}$ in. is desired;

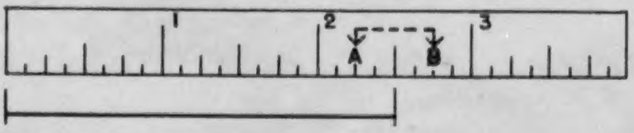
   (c) To the nearest $\frac{1}{16}$ in.

4. Marked in 16th in.

Answers to Class Discussion Exercises 7-5c

1. $\frac{1}{4}$ in. 2. $\frac{1}{16}$ in.

3.



Right end-point of segment could lie between A and B.
Greatest possible error is $\frac{1}{4}$ in.

4. $\frac{1}{8}$ in. 5. $\frac{1}{32}$ in.

6. (b) $\frac{1}{16}$", $\frac{1}{2} \times \frac{1}{16}" = \frac{1}{32}$"; (e) $\frac{1}{16}$", $\frac{1}{32}$";

   (c) $\frac{1}{8}$", $\frac{1}{16}$"; (f) $\frac{1}{8}$", $\frac{1}{16}$".

   (d) $\frac{1}{8}$", $\frac{1}{16}$";

7. (b) Between $(1\frac{15}{16} - \frac{1}{32})$ in. = $1\frac{29}{32}$ in.

   and $(1\frac{15}{16} + \frac{1}{32})$ in. = $1\frac{31}{32}$ in.

   (c) Between $(4\frac{3}{8} - \frac{1}{16})$ in. = $4\frac{5}{16}$ in.

   and $(4\frac{3}{8} + \frac{1}{16})$ in. = $4\frac{7}{16}$ in.

   (d) Between $(2\frac{6}{8} - \frac{1}{16})$ in. = $2\frac{11}{16}$ in.

   and $(2\frac{6}{8} + \frac{1}{16})$ in. = $2\frac{13}{16}$ in.

(e) Between $(3\frac{10}{16} - \frac{1}{32})$ in. = $3\frac{19}{32}$ in.

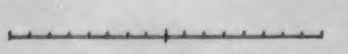
and $(3\frac{10}{16} + \frac{1}{32})$ in.

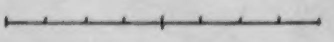
(f) Between $(7\frac{4}{8} - \frac{1}{16})$ in. = $7\frac{7}{16}$ in.

and $(7\frac{4}{8} + \frac{1}{16})$ in. = $7\frac{9}{16}$ in.

Answers to Exercises 7-5a

1. $\frac{1}{2}$

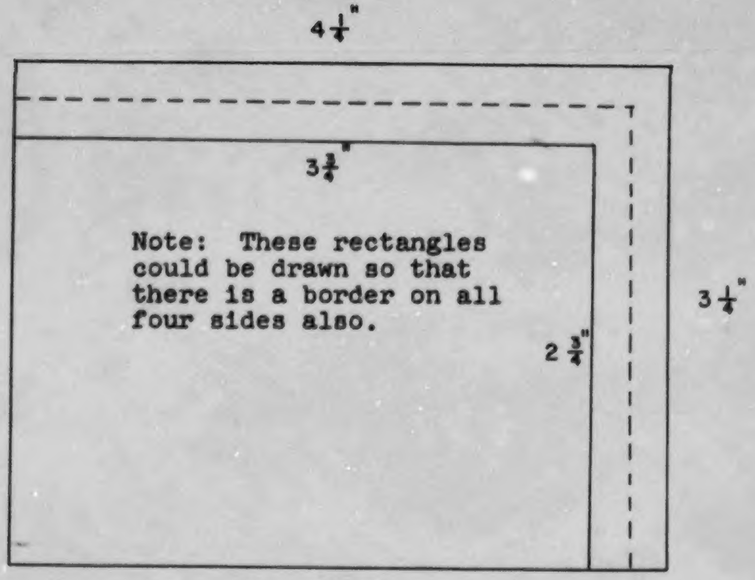
2.  (Divided into 8ths of an inch)

3.  (Divided into 4ths of an inch)

4.

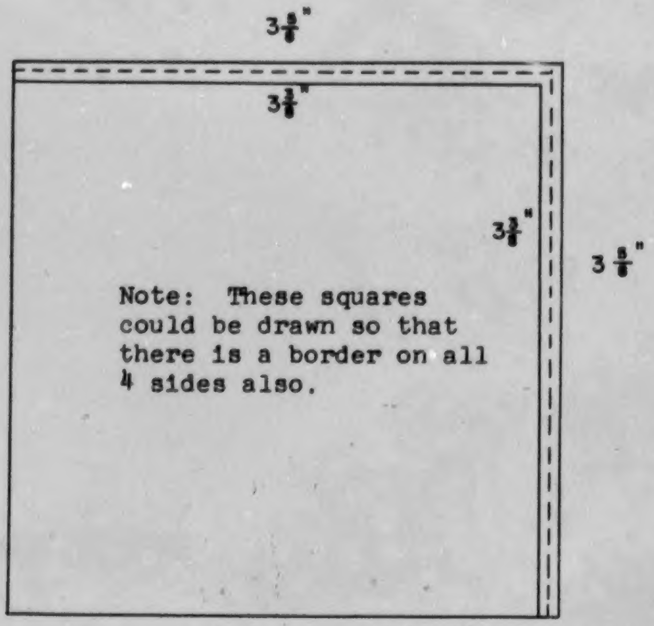
Figure		(1)		(2)		(3)	
		$\frac{1}{2}''$ Marks	G. P. Error	$\frac{1}{8}''$ Marks	G. P. Error	$\frac{1}{16}''$ Marks	G. P. Error
(a)	ℓ w	$3\frac{0}{2}''$ $1\frac{1}{2}''$	$\frac{1}{4}''$	$2\frac{8}{8}''$ or $2\frac{7}{8}''$ $1\frac{3}{8}''$	$\frac{1}{16}''$	$2\frac{15}{16}''$ $1\frac{6}{16}''$	$\frac{1}{32}''$
(b)	ℓ w	$2\frac{0}{2}''$ $2\frac{0}{2}''$	$\frac{1}{4}''$	$2\frac{0}{8}''$ or $2\frac{1}{8}''$ $2\frac{0}{8}''$ or $2\frac{1}{8}''$	$\frac{1}{16}''$	$2\frac{1}{16}''$ $2\frac{1}{16}''$	$\frac{1}{32}''$
(c)	ℓ w	$1\frac{1}{2}''$ or $2\frac{0}{2}''$ $1\frac{0}{2}''$	$\frac{1}{4}''$	$1\frac{8}{8}''$ $\frac{7}{8}''$ or $\frac{8}{8}''$	$\frac{1}{16}''$	$1\frac{15}{16}''$ $\frac{15}{16}''$	$\frac{1}{32}''$
(d)	ℓ w	$2\frac{1}{2}''$ $\frac{1}{2}''$	$\frac{1}{4}''$	$2\frac{5}{8}''$ $\frac{4}{8}''$ or $\frac{5}{8}''$	$\frac{1}{16}''$	$2\frac{10}{16}''$ $\frac{8}{16}''$	$\frac{1}{32}''$

5.


4 1/4"
3 3/4"
3 1/4"
2 3/4"

Note: These rectangles could be drawn so that there is a border on all four sides also.

6.


3 5/8"
3 3/8"
3 3/8"
3 5/8"

Note: These squares could be drawn so that there is a border on all 4 sides also.

Answers to Exercises 7-5b

1.

CD = $1\frac{3}{4}$"

2. CD = $1\frac{3}{4}$" $\pm$ $\frac{1}{8}$"

3. CD = $1\frac{3}{4}$"

4. Between $1\frac{5}{8}$" and $1\frac{7}{8}$"; $\frac{1}{8}$"

5. (a) 8^{th}

   (b) $1\frac{3}{16}$" and $1\frac{5}{16}$"

   (c) $\frac{1}{16}$"

   (d) $1\frac{1}{4} \pm \frac{1}{16}$ in.

6. (a) 8^{th}

   (b) $2\frac{3}{16}$" and $2\frac{5}{16}$"

   (c) $\frac{1}{16}$"

   (d) $2\frac{2}{8}$"

7. Answers will vary. Be sure the $\pm$ greatest possible error notation is used.

8. Answers will vary. Be sure precision is shown by not changing fractions to lower terms.

9. Note: The answers in Problems 9 and 10 are given for rulers divided into 16^{ths} of an inch.

   AB = $4\frac{3}{16}$" and $4\frac{3}{16}$" $\pm$ $\frac{1}{32}$"

   BC = $2\frac{12}{16}$" and $2\frac{3}{4}$ $\pm$ $\frac{1}{32}$" or $2\frac{11}{16}$" and $2\frac{11}{16}$ $\pm$ $\frac{1}{32}$"

   AC = $6\frac{0}{16}$" and 6 $\pm$ $\frac{1}{32}$"

10. AB = $3\frac{0}{16}$" and 3 $\pm$ $\frac{1}{32}$"

    BC = $1\frac{13}{16}$" and $1\frac{13}{16}$ $\pm$ $\frac{1}{32}$"

    CD = $5\frac{4}{16}$" and $5\frac{1}{4}$ $\pm$ $\frac{1}{32}$"

    AD = $2\frac{10}{16}$" and $2\frac{5}{8}$ $\pm$ $\frac{1}{32}$"

7-6. Measurement of Angles

In Chapter 4 the pupils have been introduced to the concept of angle as the set of points on two rays with a common endpoint, and have learned to describe the position of a point as being on the angle, in the interior of the angle, or in the exterior of the angle. The measurement of angle follows essentially the same ideas as measurement of a line segment; that is, (1) the unit for measuring an angle must be itself an angle; (2) the interior of the angle is subdivided by drawing rays which form angles like the unit angle; (3) the measure of the angle is the number of unit angles into which it is subdivided. After these ideas are developed in Exercises 7-6a, the standard unit, the degree, and the scale for measuring angles in degrees, are introduced. The unit angle is determined by a set of 181 rays drawn from the same point. These rays determine 180 congruent angles which, together with their interiors make a half-plane and the line which determines the half-plane. The rays are numbered in order from 0 to 180, forming a scale. Figure 7-6a shows such a scale. One of the 180 congruent angles is selected as the standard unit angle. Its measurement is called a degree.

The most common type of protractor shows the scale only from 0 to 180, and usually contains two such scales, one with 0 at the right end and the other with 0 at the left end of the semicircular scale. Each pupil should have a protractor and should become proficient in using it, both to measure a given angle and to draw an angle of a specified number of degrees.

The angle is defined as having a measure greater than 0 degrees and less than 180 degrees.

Angles are classified as acute, right, or obtuse, according to their measures. The pupils should practice estimating the number of degrees in an angle by comparing it with a right angle.

Perpendicular rays are defined as rays which form a right angle and the term "perpendicular" is also applied to lines, segments, and to combinations of these.

Answers to Exercises 7-6a

1. The size of $\angle RST \approx 3$ u
2. The size of $\angle ABC \approx 5$ u
3. The size of $\angle EDF \approx 4$ u
4. The size of $\angle KML \approx 6$ u

Answers to Exercises 7-6b

1. 48°
2. 95°
3. 90°
4. 80°

Answers to Exercises 7-6c

1. (a) 15
   (b) 35
   (c) 60
   (d) 90
   (e) 100
   (f) 25
   (g) 60
   (h) 145
   (i) 40
   (j) 85
   (k) 140
   (ℓ) 65
2. The size of $\angle A \approx 50^\circ$
   The size of $\angle B \approx 90^\circ$
   The size of $\angle C \approx 123^\circ$
   The size of $\angle D \approx 74^\circ$
3. The measure of $\angle BAC \approx 35$.
4.
5. (a) 180
   (b) No. The definition of angle (Ch. 4) rules out an angle of 180° because the rays may not be collinear.

Answers to Exercises 7-6d

1. (a) $\angle B$, $\angle E$

(b) $\angle A$, $\angle C$

(c) $\angle D$, $\angle F$

2\.

3. (a) 0, 90

(b) 90, 180

4. (a) Obtuse angles: $\angle BAF$, $\angle BAG$, $\angle BAH$

Acute angles: $\angle BAC$, $\angle BAD$

Right angles: $\angle BAE$

(b) Acute angles: $\angle EAD$, $\angle EAC$, $\angle EAF$, $\angle EAG$, $\angle EAH$

Right angles: $\angle EAB$, $\angle EAK$

Obtuse angles: None

(c) Right angles: $\angle KAE$

Obtuse angles: $\angle KAD$, $\angle KAC$

Acute angles: $\angle KAH$, $\angle KAG$, $\angle KAF$

5. (a) $\angle ABC$ is acute

$\angle EDF$ is a right angle

$\angle HKG$ is acute

$\angle JML$ is obtuse

$\angle PRN$ is a right angle

$\angle QTS$ is obtuse

(b) See (c)

(c) The size of $\angle ABC \approx 10^\circ$

The size of $\angle EDF \approx 90^\circ$

The size of $\angle HKG \approx 65^\circ$

The size of $\angle JML \approx 120^\circ$

The size of $\angle PRN \approx 90^\circ$

The size of $\angle QTS \approx 130^\circ$

Answers to Exercises 7-6e

1. (a) 3
   (b) 5 (3 and 1 could be so interpreted also)
   (c) 2
   (d) 1
   (e) 6
   (f) 4
2. Edges of book, edges of intersecting sidewalks, intersecting edges on a rectangular prism, etc.
3. Edges of gable roof, diagonal parking line and curb, hands of clock in many positions, etc.

Sample Questions for Chapter 7

Select from this list. There are too many questions for one test.

I. True-False Questions

(F) 1. The counting numbers are all that are needed for both counting and measuring.

(F) 2. If two simple closed curves have the same length, then the corresponding closed regions have the same area.

(T) 3. Perpendicular lines always meet so that rays with the endpoints at the intersection form right angles.

(T) 4. A rectangular closed region can be used to measure a square closed region.

(T) 5. All measuring units can be sub-divided.

(T) 6. A ruler is one application of a number line.

(T) 7. Base 2 is related to the subdivisions of an inch on the ruler commonly used in school.

(F) 8. The measure in degrees of an obtuse angle is smaller than that of a right angle.

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(T) 9. A measuring unit may have any size we choose.

(F) 10. If two rectangular closed regions have the same area, then their lengths are the same and so are their widths.

II. Multiple Choice, Completion and Matching

(c) 1. To measure a line segment, you must use as a unit:

- (a) An inch
- (b) A foot
- (c) A line segment
- (d) A square inch
- (e) None of these

(a) 2. Choose the best way to complete the statement, Standard units of measurement are used because:

- (a) It is important for people to use the same unit in dealing with each other.
- (b) Standard units give more accurate measurements than units which are not standard.
- (c) People have always used them.
- (d) They are all related to base 10 numeration.
- (e) None of these.

(e) 3. All rectangles with perimeters of 20 inches:

- (a) Have the same area.
- (b) Have areas that increase as the base increases in length.
- (c) Have areas that increase as the height increases in length.
- (d) Have the same area as a square with a 20 inch perimeter.
- (e) None of these.

(c) 4. The markings on a ruler divide each inch into 8 equal parts. The correct ways to report one measurement made with this ruler are:

(a) $3\frac{4}{16}$ in. and $(3\frac{4}{16} \pm \frac{1}{32})$ in.

(b) $3\frac{2}{8}$ in. and $(3\frac{1}{4} \pm \frac{1}{8})$ in.

(c) $3\frac{2}{8}$ in. and $(3\frac{1}{4} \pm \frac{1}{16})$ in.

(d) $3\frac{2}{8}$ in. and $(3\frac{2}{8} \pm \frac{1}{8})$ in.

(e) None of these.

(b) 5. The rays on the complete scale for measuring angles are numbered

(a) From 1° to 180° (c) None of these.

(b) From 0° to 180°

(b), (c) and (d) should be underlined. 6. Below is a list of things, some of which can be counted while others must be measured. Underline those which must be measured.

(a) Crowd at a meeting

(b) Time it takes to walk home

(c) Weight of a brick

(d) Capacity of a gas tank

(e) Coats in your closet

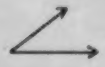
$3\frac{3}{8}$ and $3\frac{5}{8}$ 7. If a length is to be reported as $3\frac{2}{4}$ in., the true length must be between ______in. and ______in.

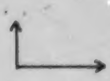
90 and 180 8. An obtuse angle is one whose measure in degrees is between ______ and ______.

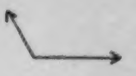
2 9. If the length and width of a rectangle are doubled, the perimeter of the new rectangle is ______times that of the original one.

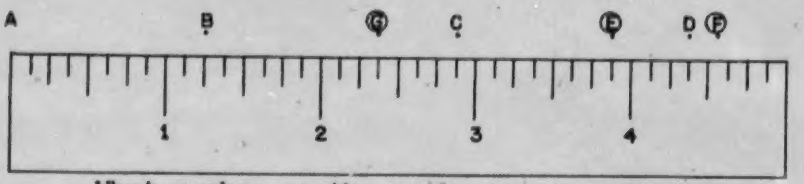
Protractor 10. The instrument used to measure an angle is called a ________

Kind or nature 11. A unit of measurement must be of the same ________ as the object measured.

Acute 12.  represents a(n) ________ angle.

Right  represents a(n) ________ angle.

Obtuse  represents a(n) ________ angle.

13. 

What number on the scale corresponds to:

$1\frac{1}{4}$" (a) Point B ________

$2\frac{7}{8}$" (b) Point C ________

$4\frac{3}{8}$" (c) Point D ________

III. Problems

1. On the number scale for Question 13, mark the point which corresponds to the numbers given below and label them with the letter indicated.

(a) $3\frac{7}{8}$ E (c) $2\frac{3}{8}$ G

(b) $4\frac{9}{16}$ F

2. Measure these segments to the nearest 16th inch. Report the results so that the correct precision is indicated.

$3\frac{4}{16}$" or (a)

$3\frac{1}{4}$" $\pm$ $\frac{1}{32}$"

$4\frac{0}{16}$" or (b)

4" $\pm$ $\frac{1}{32}$"

$1\frac{8}{16}$" or (c)

$1\frac{1}{2}$" $\pm$ $\frac{1}{32}$"

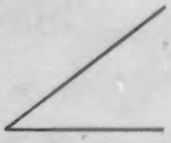
3. Measure each of these angles.

(a) 38°

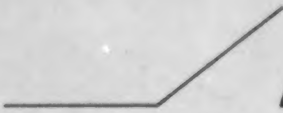
(b) 142°

(c) 74°

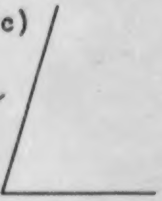
(a)



(b)



(c)



Chapter 8

AREA, VOLUME, WEIGHT AND TIME

8-1. Rectangle

The two primary aims of this section are to develop the pupils' space perception, especially as it relates to perimeters and areas, and to develop methods of finding these quantities for a rectangle by computation when the length and width are known. In development of space perception and a geometric feeling for size, it is important that models be constantly in sight. Perhaps drawings of the actual sizes of the inch square, foot square, and yard square could be kept on the blackboard as soon as area has been introduced. Possibly models could be placed on the bulletin board. Also emphasis on estimating lengths and areas will help to develop a feeling for size. As often as possible have estimates made of lengths and areas and then have them actually measured (or computed) to check the accuracy of the estimate. This emphasis will necessarily come largely from the teacher, as such questions can only be suggested occasionally in the text. It is interesting that most people are quite surprised to see the actual size of a square yard.

Two points of terminology should be clarified.

1. By its definition a rectangle is a collection of segments. Such a set of points does not properly have an area (or possibly it should be said to have area zero). The area with which we are concerned is the area of the closed rectangular region; that is, the union of the set of points on the rectangle and the set of points in its interior. In order to emphasize this, it is the plan in the text to state that we mean the area of the closed rectangular region when we say the area of the rectangle. While the student may lapse into the mathematical slang of speaking of the area of a rectangle, from time-to-time remind him of the fact that he is using mathematical slang.

The concept of area of a closed region is close to the intuitive concept which young students have. Moreover, with this idea length, area, and volume can be treated similarly in terms of unions:

1. A line segment 2 inches long is the union of two unit line segments each 1 inch in length.
2. A closed rectangular region of length 2 in. and width 1 in. is the union of two unit rectangular closed regions 1 in. by 1 in.
3. A rectangular solid 2 in. by 1 in. by 1 in. is the union of two unit cubical solids.

This closed region viewpoint is different from that used in the 1959 edition. In that edition area was based on the concept of region as defined in Chapter 4; you thought, for example, in terms of "the area of the interior of a rectangle." In terms of the "interior concept," the preceding statements 1, 2, and 3 do not hold. Many people believe that the "closed region concept" will be easier to teach because it is closer to the student's "experience."

No definitions of length, area, or volume appear in this chapter. The viewpoint is quite intuitive. When precise definitions of these words are given it can be shown that area of the interior of a simple closed curve has meaning. Moreover, this latter area is equal to the area of the corresponding closed region. In a similar fashion we could talk about the set of points on a line segment not including the endpoints as the "interior" of the line segment. This set of points would have a length equal to the length of the line segment. Similar comments can be made for volume.

Mathematically speaking, neither of these viewpoints is preferable to the other. As stated, the student has in mind the concept of a line segment. The closed region concept ties into this concept of length.

Both terminologies appear in problems. It is natural to think of the volume of a room as meaning the volume of the interior of the rectangular prism.

2. It has been emphasized in previous sections that the measure of a quantity is the number of units it contains. However, in describing the result of a measurement it is meaningless to give the number without specifying the unit. Thus when we speak of a length, or an area, or a volume we shall mean the number together with the unit, as a length of 5 ft. or an area of 5 square inches. On the other hand we should keep clearly in mind that wherever literal symbols are introduced, they stand only for numbers. We add and multiply numbers, not units. Thus if we consider a rectangle whose length is 5 in. and whose width is 3 in., and if we wish to use the notation of the number sentences $p = 2(\ell + w)$ and $A = \ell w$, we write $\ell = 5$, $w = 3$, $p = 2(5 + 3) = 16$, $A = 5 \cdot 3 = 15$ so we conclude that the perimeter is 16 in. and the area is 15 sq. in. Note that we do not write $\ell = 5$ in. or $A = 15$ sq. in. Here again, the student may in the future, without permanent harm to his character, write such mathematical slang as $A = 15$ sq. in., but at this point it has seemed better to distinguish clearly between a length, which requires specifying a unit, and the number of units ℓ in this length.

The question about the relation of squares to rectangles in the first paragraph of this section has two purposes. The discussion should first serve to clarify any doubt as to exactly what a square is. Then, bringing out that such a figure is a special case of a rectangle gives a good example of one set contained in another. The question in the second paragraph is intended to bring out that a segment is measured by a number line of segments and our model of this is a ruler. However, a closed rectangular region not being a segment, cannot be measured by segments, but must be measured by a closed rectangular region--something of the same kind. Some pupils may argue that you could measure the closed rectangular region with a ruler by seeing how many times the ruler could be fitted on it. In a sense this is true, but point out that, if you do that, you are no longer considering the ruler as a model of a number line and using just its numbered edge. You would actually be using it as a closed rectangular region.

Exercises 8-1a

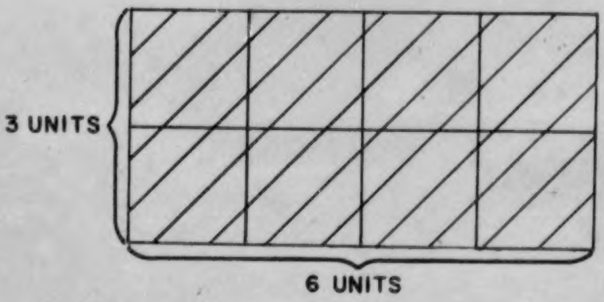
Problems 1-5 are part of the developmental work and should definitely be used, although some or all of them might well be done in class. The other problems in the set are practice exercises on understanding perimeter, computation, and change of units. Problem 10 should be included in the assigned work as these ideas often seem troublesome.

Areas of Rectangles

The question in the first paragraph about an instrument to use directly in measuring areas should provoke useful discussion. Someone may well point out that you could have a network of closed square regions (or whatever figure you use as unit of area) marked on transparent plastic and just place this on the rectangular closed region being measured. This is quite true, and is both an interesting and valuable idea. However, it should be pointed out that in using this you would actually have to count up the squares (and parts of squares). You cannot number the squares ahead of time so that you can just look at the number attached to a "last" square and say that is (approximately) the desired area. The difficulty is that while all segments have the same shape in the sense of being fitted by a number line (ruler), areas may occur in all sorts of shapes so that no prearranged pattern can be designed to fit all the shapes to be measured in such a way as to read off areas at sight. Your pupils may be interested to know that there are instruments which can be run around the boundaries of a simple closed curve and give at once an approximate value for the area of the closed region. One such instrument is the planimeter, but its theory is far beyond the level of our work here.

In developing the method for computing areas of rectangles from the length and width it is noted that different ways of counting the unit areas illustrate the commutative property of multiplication.

It is hoped that class discussion will bring out the advantage of having a unit of area which is a closed square region, one unit of length on a side. In the case of the illustration of the rectangle 6 units by 3 units, notice the effect of using, as unit of area, a closed square region $1\frac{1}{2}$ units on a side. There is no trouble about covering the closed rectangular region with these units as shown below.



However, the number of squares in each row is no longer the number of linear units in the length (6), and the number of rows is no longer the number of linear units in the width (3). Thus we would lose the relationship of finding the number of square units of area by multiplying the numbers of linear units in the length and width. It certainly would be possible to devise methods of computing the number of these new square units of area, but it would also be more complicated.

Class Exercises 8-1a

Problems 1-3 are developmental, leading to the method for computing area of a rectangle from the numbers of linear units in the length and width.

Exercises 8-1b

In connection with Problem 3 notice the comments in the first paragraph of the commentary in the section about emphasizing space perception. Estimating areas of doors, windows, blackboards, etc. in the various units of area and then measuring to check estimates is very valuable experience. Following Problem 3, which furnishes the information necessary, many of the problems call for conversion between various units of area. The process is the same as for linear units, but it may be well to illustrate it for the class. For example, since 1 sq. ft. = 144 sq. in., then 1 sq. in. = $\frac{1}{144}$ sq. ft. Thus to change 360 sq. in. to square feet we write 360 sq. in. = 360 · 1 sq. in = 360 · $\frac{1}{144}$ sq. ft. = $\frac{360}{144}$ sq. ft. = $2\frac{1}{2}$ sq. ft.

Problems 4, 5 are designed to clarify the common confusion between the 3-inch square and the area of 3 square inches, and to emphasize the different possible shapes an area of 1 square inch may take. Problems 11-15 are a connected group and are extremely important. They emphasize the effect of doubling the dimensions of a rectangle either separately or simultaneously. Problems 11-14 approach the problem geometrically and it should be emphasized that such doubling amounts to laying identical rectangles end to end or side by side. The dotted lines in the figures in the answer key indicate the relationships we hope the pupil will see. Problem 15 shows how these geometric facts are related to properties of rational numbers that have already been studied.

Exercises 8-1c

Problems 1-5 are drill problems in finding areas and perimeters and in converting units, with emphasis that such answers must generally be considered approximate. This section could provide further opportunity for estimating and verification. Problem 7 illustrates finding areas as a difference of known areas.

Class Exercises 8-1b

Problems 1-3 are a unit which develop the idea of precision and greatest possible error in a computed measurement. They are important if the class is capable, though not easy. The length and width of a rectangle are given with a precision of $\frac{1}{4}$ inch (i.e. measurements to the nearest quarter inch). The pupil is asked to compute the area from these figures. This proves to be $8\frac{15}{16}$ sq. in. He then draws the largest and smallest rectangles which are represented by the given measurements with this precision and computes their areas. The largest possible difference between the true area and the computed area above proves to be $\frac{49}{64}$ sq. in. so the result is written in the form

$$\text{Area is } (8\tfrac{15}{16} \pm \tfrac{49}{64}) \text{ sq. in.}$$

The greatest possible error is $\frac{49}{64}$ sq. in. and the precision, which is the total range about $8\frac{15}{16}$ within which the true answer must occur, is $\frac{49}{32}$ sq. in. (i.e. twice the greatest possible error). The important thing to emphasize is the size of the possible error in the calculated answer. The average pupil who computes an area as above and gets $8\frac{15}{16}$ sq. in. is inclined to assume this is correct to the nearest sixteenth of a square inch, but the calculation shows that the possible error is much greater than this.

The intention in these problems is to promote a healthy skepticism about the precision of calculated answers, i.e. to have the students realize that working with approximate figures not only gives approximate results but that the errors may be greatly expanded and that the results are generally not as good as they look. As a matter of fact they are practically never as good as the form of the answer suggests.

Answers to Exercises 8-1a

3. Equal

4. The other two sides are then 6 inches and 4 inches long, and the perimeter is 20 inches.

5. The number of units in the other two sides are ℓ and w. The number sentence could be any of the following:
$p = \ell + w + \ell + w$ or $p = 2\ell + 2w$ or $p = 2(\ell + w)$

6. 1200 ft.; 400 yds.

7. $2000.

8. 600 in. or 50 ft. or $16\frac{2}{3}$ yds. (if inches, feet, yards are units chosen).

9. 456 in. or 38 ft. or $12\frac{2}{3}$ yds. Does not matter where doorways are.

10. 12 ft. by 12 ft. - no; not enough fence.
8 ft. by 3 ft. - no; doesn't use all the fence.
8 ft. by 4 ft. - yes.
Any five of the following: 11 ft. by 1 ft., 10 ft. by 2 ft., 9 ft. by 3 ft., 8 ft. by 4 ft., 7 ft. by 5 ft., 6 ft. by 6 ft.

11. $19\frac{4}{9}$ yd.

12. Any two of: $2\frac{3}{4}$ mi., 4840 yds., 14,520 ft., 174,240 in.

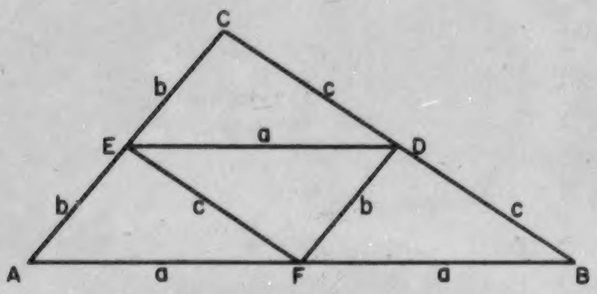
[pages 299-301]

13. $687.50

14. The man from T saved nothing; the man from B saved 2 blocks.

*15. Side opposite given side is 40 ft. long, other two sides each 80 ft. long; either $2x + 2 \cdot 40 = 240$ or $2(x + 40) = 240$.

*16.



Simple closed curve	Numbers of units in perimeter
AFE	a + b + c
FBD	a + b + c
EDC	a + b + c
EFD	a + b + c
AFDE	2a + 2b
EFBD	2a + 2c
EFDC	2b + 2c
ABDE	3a + b + c
FBCE	a + b + 3c
DCAF	a + 3b + c
ABC	2a + 2b + 2c

Answers to Class Exercises 8-1a

1. 11; 5; 55; $\frac{1}{16}$ sq. in.; $\frac{55}{16}$ sq. in. or $3\frac{7}{16}$ sq. in. The numbers 11 and 5 are the numbers of quarter inches in the length and width. Thus they are the numerators when the length of $2\frac{3}{4}$ is written as $\frac{11}{4}$ and when the width of $1\frac{1}{4}$ is written as $\frac{5}{4}$.

2. Small square $\frac{1}{2}$ inch by $\frac{1}{2}$ inch; 11; 9; $\frac{99}{4}$ or $24\frac{3}{4}$.

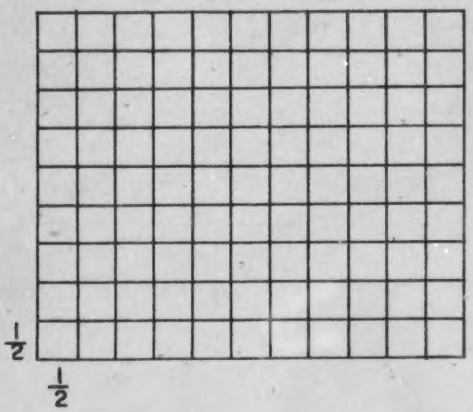
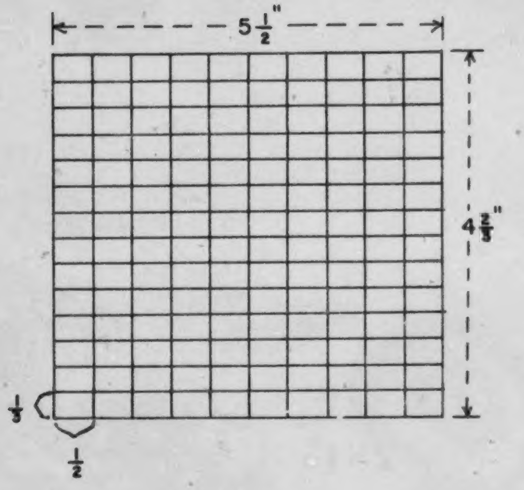


Figure not necessary to do problem. Use method of Problem 1 above.

*3.



The division of the unit squares is suggested by the fractions in which measurements are given. Each closed rectangular region is = $\frac{1}{6}$ sq. in. There are $11 \times 14 = 154$ such regions, so the area is $\frac{154}{6}$ sq. in. Note that $\frac{154}{6}$ is exactly $\frac{11}{2} \cdot \frac{14}{3}$.

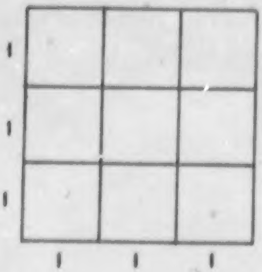
Answers to Exercises 8-1b

1. The number of square units of area in the interior of a rectangle is the product of the number of units in the length by the number of units in the width.

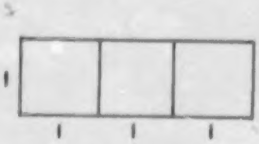
2. $A = \ell w$

3. 1 square foot contains 144 square inches; 1 square yard contains 9 square feet.

4.



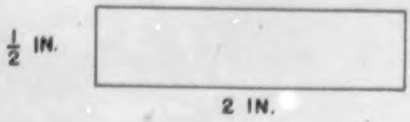
Three-inch square.

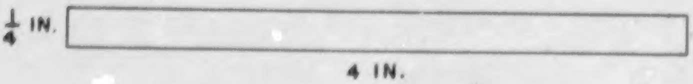


An area of three square inches.

The 3 inch square is larger. Its area is 9 square inches (These figures are not full size but to a scale with $\frac{1}{2}$ inch representing an inch.)

5.





6. 108 sq. ft., 12 sq. yds., 15,552 sq. in.

7. 8100 sq. ft., 900 sq. yds.

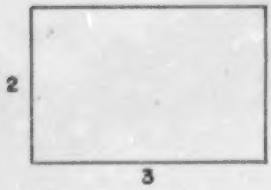
8. Area is 3600 sq. ft. This is less than half the area of a baseball diamond. Actually it is $\frac{4}{9}$ of it.

9. 3,097,600 sq. yds. in 1 sq. mi.

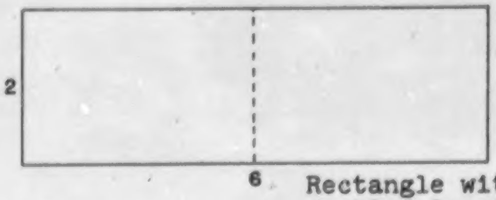
10. Area of smaller inside rectangle is $2 \cdot 3 = 6$ square units.
Area of larger inside rectangle is $6 \cdot 3 = 18$ square units.
Area of outside rectangle is $8 \cdot 3 = 24$ square units.

$$24 = 6 + 18$$
$$8 \cdot 3 = 2 \cdot 3 + 6 \cdot 3$$
$$8 \cdot 3 = (2 + 6)3 \qquad \text{Distributive property}$$

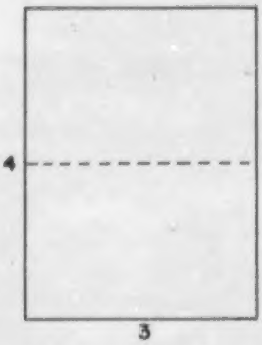
11.



Original rectangle.



Rectangle with doubled length.

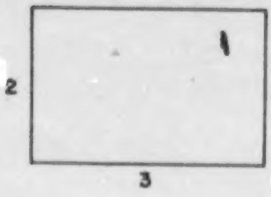


Rectangle with doubled height.

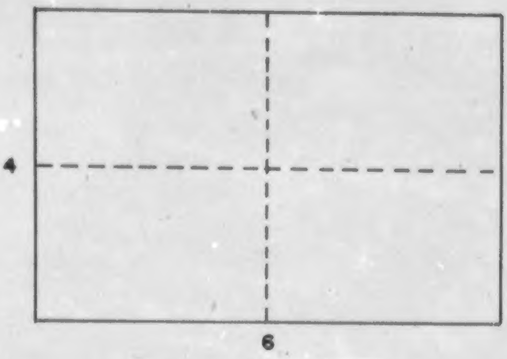
Original area is 6 square units. Both new areas are 12 square units. New areas are double the original one. The dotted lines are drawn to show geometrically why this is true. They are not part of the drawing asked for in the problem.

12. No. Doubling one measurement (length or width) of any rectangle doubles its area.

13.



Original rectangle.



New rectangle.

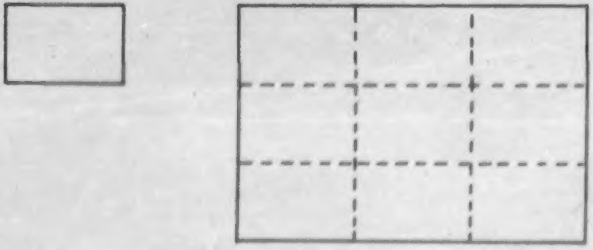
(Dotted lines not a part of required drawing, but are shown to make clear the geometric relationship). Doubling both measurements (length and width) of any rectangle multiplies the old area by 4.

14. Perimeter of original rectangle is 10 units. Perimeter of new rectangle is 20 units. Doubling both measurements (length and width) of any rectangle doubles its perimeter. (The drawing of Problem 13 makes clear that this relation does not depend on the particular measurements.)

*15. $626 \cdot 422 = 2 \cdot 313 \cdot 2 \cdot 211 = 2 \cdot 2 \cdot 313 \cdot 211 =$
$2^2 \cdot 313 \cdot 211$

Commutative and associative properties are used. The statement says the area of the new rectangle is 4 times that of the original rectangle $(4 = 2^2)$. Yes, it agrees with Problem 13.

*16. If length and width are tripled, the area of the new rectangle is 9 times that of the original rectangle. $(9 = 3^2)$.



If length is doubled and width tripled, the area is multiplied by 6. $(6 = 2 \cdot 3)$

Answers to Exercises 8-1c

2. Area ≈ 2520 sq. in., $17\frac{1}{2}$ sq. ft., $1\frac{17}{18}$ sq. yd.
   Perimeter ≈ 204 in., 17 ft., $5\frac{2}{3}$ yd.
3. Area ≈ $\frac{3}{25}$ sq. mi., 371,712 sq. yd.
4. 14 lbs.
5. $2\frac{1}{2}$ sq. yds.
6. 99 sq. ft., (or 11 sq. yds.)
7. 570 sq. ft., $63\frac{1}{3}$ sq. yds. (or 82,080 sq. in.).
   It does not matter where the opening is.
8. By Problem 14 of Exercises 8-1b the perimeter will be doubled by doubling length and width, so the order for the fence is correct. By Problem 13 of Exercises 8-1b the area of the garden will be multiplied by 4, so the fertilizer order is wrong. He should order 4 bags of fertilizer.

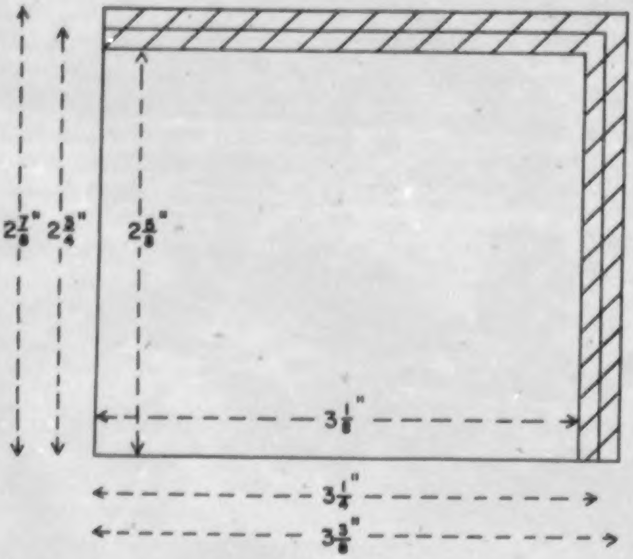
Answers to Class Exercises 8-1b

1. $A = \ell w$, $A \approx \left(\frac{13}{4}\right)\left(\frac{11}{4}\right)$, $A \approx 8\frac{15}{16}$ so the area is approximately $8\frac{15}{16}$ sq. in. For figure see Problem 2.

2. (a) $3\frac{1}{8}$ and $3\frac{3}{8}$

   (b) $2\frac{5}{8}$ and $2\frac{7}{8}$

   (c) and (d)



3. Area of smallest rectangle is $8\frac{13}{64}$ sq. in.

   Area of largest rectangle is $9\frac{45}{64}$ sq. in.

   Difference between answer to Problem 9 and smallest possible true answer is $\frac{47}{64}$ sq. in.

   Difference between answer to Problem 9 and largest possible true answer is $\frac{49}{64}$ sq. in.

   Greatest possible error is therefore $\frac{49}{64}$ sq. in.

   The area is $\left(8\frac{15}{16} \pm \frac{49}{64}\right)$ sq. in.

Answers to Exercises 8-1d

1. (1)

Minimum Rectangle	Measured Rectangle	Maximum Rectangle	
$3\frac{1}{4}$ in.	$3\frac{1}{2}$ in.	$3\frac{3}{4}$ in.	Length
$4\frac{1}{4}$ in.	$4\frac{1}{2}$ in.	$4\frac{3}{4}$ in.	Width
$\frac{221}{16}$ sq. in. or $13\frac{13}{16}$ sq. in.	$\frac{252}{16}$ sq. in. or $15\frac{3}{4}$ sq. in.	$\frac{285}{16}$ sq. in. or $17\frac{13}{16}$ sq. in.	Area

(2) Calculated measured area - calculated minimum area = difference.

$$\frac{252}{16} - \frac{221}{16} = \frac{31}{16}.$$

Calculated maximum area - calculated measured area = greatest possible error.

$$\frac{285}{16} - \frac{252}{16} = \frac{33}{16}.$$

(3) Thus the precision of the calculated area is

$$15\frac{3}{4} \pm \frac{33}{16} \text{ sq. in.}$$

2. (1)

Minimum Rectangle	Measured Rectangle	Maximum Rectangle	
$1\frac{5}{8}$ in.	$1\frac{3}{4}$ in.	$1\frac{7}{8}$ in.	Length
$2\frac{1}{8}$ in.	$2\frac{1}{4}$ in.	$2\frac{3}{8}$ in.	Width
$\frac{221}{64}$ sq. in. or $3\frac{29}{64}$ sq. in.	$\frac{63}{16}$ sq. in. or $3\frac{15}{16}$ sq. in.	$\frac{285}{64}$ sq. in. or $4\frac{29}{64}$ sq. in.	Area

(2) Calculated measured area - calculated minimum area = difference.

$$\frac{252}{64} - \frac{221}{64} = \frac{31}{64}$$

Calculated maximum area - calculated measured area = greatest possible error.

$$\frac{285}{64} - \frac{252}{64} = \frac{33}{64}$$

(3) Thus the precision of the calculated area is:

$$\left(3\frac{15}{16} \pm \frac{33}{64}\right) \text{sq. in.}$$

3. (1)

Minimum Rectangle	Measured Rectangle	Maximum Rectangle	
$2\frac{5}{16}$ in.	$2\frac{3}{8}$ in.	$2\frac{7}{16}$ in.	Length
$3\frac{7}{16}$	$3\frac{4}{8}$ in.	$3\frac{9}{16}$ in.	Width
$\frac{2035}{256}$ sq. in. or $7\frac{243}{256}$ sq. in.	$\frac{532}{64}$ sq. in. or $8\frac{5}{16}$ sq. in.	$\frac{2223}{256}$ sq. in. or $8\frac{175}{256}$ sq. in.	Height

(2) Calculated measured area - calculated minimum area = difference.

$$\frac{2128}{256} - \frac{2035}{256} = \frac{93}{256}$$

Calculated maximum area - calculated measured area = greatest possible error.

$$\frac{2223}{256} - \frac{2128}{256} = \frac{95}{256}$$

(3) Thus the precision of the calculated area is

$$\left(8\frac{5}{16} \pm \frac{95}{256}\right) \text{sq. in.}$$

Answers to Exercises 8-1e

1. (10)(10) or 100. There are 100 sq. millimeters in 1 sq. centimeter.

2. (100)(100) = 10,000. There are 10,000 square centimeters in 1 square meter.

3. (100)(10,000) = 1,000,000. There are 1,000,000 square millimeters in 1 square meter.

4. The area of a 3 centimeter square is 9 sq. centimeters. It is therefore larger than the area of a rectangle whose area is 3 square centimeters.

5. A = (2)(3) = 6. The area is 6 square meters.
   2(2 + 3) = 10. The perimeter is 10 meters.

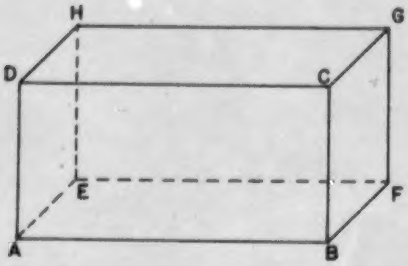
6. (4)(3) - (1)(1) = 12 - 1 = 11. There are 11 sq. meters of floor space.

8-2. Rectangular Prism

The remarks under Section 8-1 about development of space perception should be repeated here emphatically. The concept of volume and the sizes of different units of volume need much reinforcing. Models of cubic inches will be constructed but models of cubic feet and cubic yards are necessary also. Practice in estimating volumes should be very helpful. A framework for a model of a cubic yard can be formed from 12 yardsticks. Most adults are astounded at the size of a cubic yard when they see one. One teacher had students bring in cardboard cartons that could be cut to the size of a cubic foot and assembled a model cubic yard from 27 of these boxes. Grocery cartons which are partitioned by cardboard dividers may be helpful in visualizing the subdivision of a volume into units.

Just as a rectangle is composed of the points on its segments, so a rectangular prism is made up of the points on its faces, i.e. on its surface. Thus if a brick is suggested as an example of this figure, point out that the prism consists only of the surface of the brick. A rectangular solid consists of the points on the surface of a rectangular prism together with those in its interior. For this reason though we do speak of the volume of a rectangular prism, we actually mean the volume of the rectangular solid. As in the last section, the stating of a measurement requires both the number and the unit used, but any letter in a number sentence stands only for a number.

The rectangular prism with its 6 faces, 12 edges, and 8 vertices will need to be shown to the class with models and illustrations. The approach is frankly intuitive. However, once it has been agreed that the faces are rectangles, this can be used to deduce the fact that opposite faces have the same measurements. For example in the following figure



$\overline{EF}$ and $\overline{AB}$ have the same length because they are opposite sides of rectangle ABFE.

$\overline{AB}$ and $\overline{CD}$ have the same length because they are opposite sides of rectangle ABCD.

$\overline{CD}$ and $\overline{GH}$ have the same length because they are opposite sides of rectangle DCGH.

Thus the four segments $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, $\overline{GH}$ all have the same length. Similarly $\overline{AE}$, $\overline{DH}$, $\overline{CG}$ and $\overline{BF}$ have the same length and $\overline{AD}$, $\overline{EH}$, $\overline{FG}$, and $\overline{BC}$ have the same length. This shows that any two opposite faces have the same measurements.

Exercises 8-2a

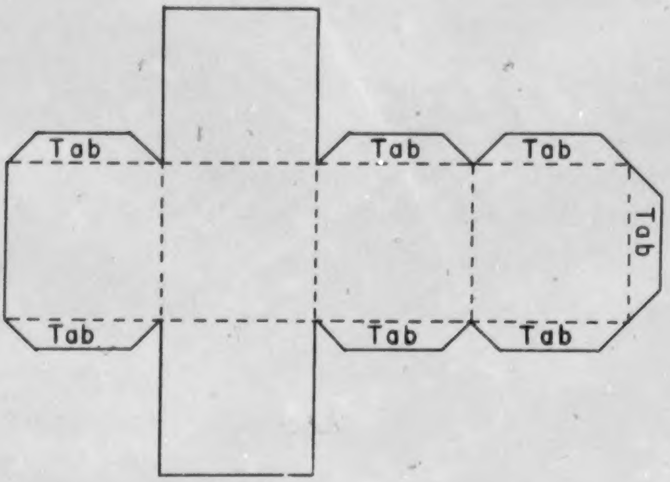
These problems will probably be difficult, not because of the arithmetic, which is easy, but because of the spacial visualizing required. It is suggested that in doing these problems each pupil keep a box or other model in front of him to help visualize the problems that are presented. Except for the difficulty of visualizing, the problems are applications of work on rectangles in the last section. There is further drill on change of units. Problems 3 and 6 give further practice on finding areas by subtraction. Problem 9 introduces the term cube.

In discussing the choice of a unit of volume bring out that its edges are units of length and its faces units of area, while its interior is the unit of volume.

Class Exercises 8-2

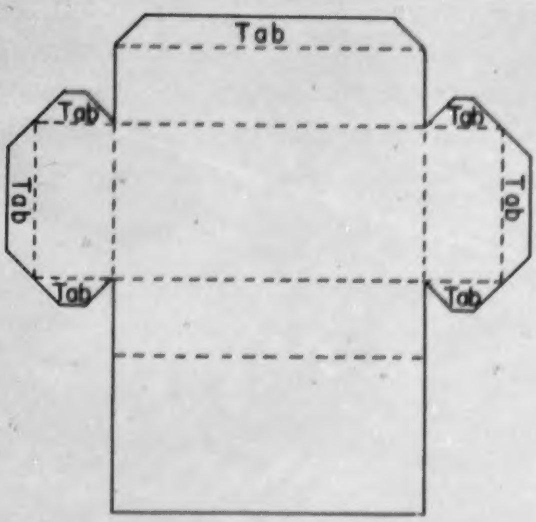
A pattern for a 1 inch cube is given here. These cubes should be made in advance so that they are ready for use in the classroom. The problems of the set should be done in class. It is extremely important that the pupil see and be able to visualize that the effect of doubling the length of a rectangular solid amounts to laying two such solids end to end, both just like the original one. Similarly, that doubling two of the measurements amounts to putting four such solids together and that doubling all three measurements

Pattern for 1 inch Cube



[pages 320-322]

Pattern for Model of Problem 7, Exercises 8-2c



is equivalent to putting eight such cubical solids together. It is this visual perception, not a counting up of the cubical solids used, that is really significant. Also the models formed here show nicely the layers of solids which are used in the next development.

In the question asked just before Exercises 8-2b, observe that if the base contains a half-unit square, we need only slice a unit cube in two vertically and stand this half unit cube on the half unit square. Similarly, for a quarter unit square and so on. Thus the total number of cubical solids in the bottom layer of volume is always the same as the number of square units in the base.

Exercises 8-2b

These problems have all been designed to emphasize the relation symbolized by the number sentence $V = Bh$. They have been deliberately chosen in such a way that the actual shape of the base is not known in any of them. This is to lay the foundation for the discussion later of volumes of prisms and cylinders, as well as the specific case of the rectangular prism which is discussed next.

In the question asked just before Exercises 8-2c the properties involved are the commutative and associative properties of multiplication.

Exercises 8-2c

Problems 3 and 4 are an integral part of the development. Problem 5 lays the basis for changes of cubic units which are introduced frequently thereafter. Problem 7 is to clarify confusion of a 3-inch cube with a volume of 3 cubic inches. Problem 8 emphasizes two different possible shapes of a volume of 1 cubic inch. Patterns for the models, if this is necessary, are found on Pages 236-37. Problem 9 is an algebraic approach to the geometric ideas developed in Class Exercises 8-2. Problem 10 affords opportunity for more estimating and measuring. Problems 18-20 develop the ideas of greatest possible error and precision for computed volumes. They are comparable to Problems 1-3 of Exercises 8-1b and the same comments apply. With respect to error in computed volume, it is possible to give a graphic illustration of the substantial effect on volume of a fairly small error in measurements of length, width, and height as follows: Take a rectangular block of cheese (a two-pound package possibly, though a smaller one would do) and imagine an error of $\frac{1}{8}$ inch. With a cheese cutter slice off $\frac{1}{8}$ inch thick slices from three faces which meet at a vertex. The total volume of these slices is the change in volume due to an error of only $\frac{1}{8}$ inch in each of the three measurements. This is quite an impressive amount. A rectangular block of clay could perhaps be used for the same purpose. This demonstration could be done without attempting to do Problems 18-20 above. As noted earlier, a skepticism with respect to the precision of such computed answers is a healthy thing.

In the discussion of dimension some of your better pupils may raise the question of describing the location of the sugar in other ways than by motions parallel to the edges of a room. For example in the second figure in the section under Dimension it may be suggested that the fly at A might simply point out the direction of S and tell his friend to crawl a certain distance in that direction. This is an excellent idea. However, note that these directions still call for two numbers, one describing the angle telling the direction in which the fly is to crawl, and the other giving the distance he must crawl. A precise definition of dimension involves very substantial difficulties beyond the scope of this course, but you will find that any "reasonable" way of describing location of points in the different sets will use the same number of numbers in the description, so that concept of dimension has meaning.

In the determination of volume for an irregular stone you may get various interesting suggestions to consider. One method is the immersion method, which would work as follows: Take a rectangular prism partially filled with water (or sand or salt or some other convenient substance). By measuring the length and width of the container and the depth of the water, the volume of water can be found. Now immerse the stone and determine the new volume. The difference of the volumes is the volume of the stone. In practice it is often a problem to find a suitable rectangular container. Sometimes toothbrush containers come in this shape and would do for small stones. Rectangular aquariums are good if you can find one and want to measure a good sized stone. If salt or sand is used as material, plenty of cardboard boxes are available.

[pages 325-329]

In the experiments to find the number of cubic inches in a liquid pint or in a bushel, the theoretical volume of the liquid pint is $28\frac{7}{8}$ cu. in. and a bushel is approximately 2150 cu. in. Possibly the bushel problem could be done by a few students as an outside project.

Answers to Exercises 8-2a

1. 52 square units
2. 152 square inches
3. 230 sq. ft., $25\frac{5}{9}$ sq. yd.
4. Area of glass is 200 sq. in. = $1\frac{7}{18}$ sq. ft.
   Area of wood is 1100 sq. in. = $7\frac{23}{36}$ sq. ft.
5. (a) 180 squares of tile
   (b) 720 squares of tile
6. 396 sq. ft. No, it does not matter where windows are placed.
7. 3 quarts
8. 312 in., 26 ft., $8\frac{2}{3}$ yds.
9. 1944 sq. in., $13\frac{1}{2}$ sq. ft.

*10. $S = 2\ell w + 2wh + 2\ell h$ or $S = 2(\ell w + wh + \ell h)$

11. $e = 4\ell + 4w + 4h$ or $e = 4(\ell + w + h)$
12. There are 6 sq. ft. of surface to varnish on each box, so 300 sq. ft. to varnish in all. One pint of varnish will cover 72 sq. ft., so 2 qts. will cover 4 times as much or 288 sq. ft. This is not enough, so answer to problem is no.

Answers to Class Exercises 8-2

1. Cubic foot - cubical solid each edge of which is 1 ft.
   Cubic yard - cubical solid each edge of which is 1 yard.
   Cubic meter - cubical solid each edge of which is 1 meter.

2. 24

3. The number of cubes for each of the new solids is just double that in the original solid.

4. Doubling one measurement (length, width, or height) of any rectangular prism doubles the volume of its interior.

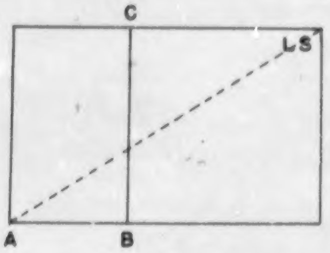
5. Ratio of number of cubes in new solid to number in original solid is $\frac{4}{1}$. That is, there are 4 times as many. The result would be the same if any two of the measurements are doubled.

6. Doubling any two measurements of any rectangular prism quadruples its volume (i.e. multiplies it by 4).

7. Ratio is $\frac{8}{1}$. That is, there are eight times as many.

8. Doubling all three measurements of any rectangular prism yields a prism whose volume is 8 times that of the original prism.

9. BRAINBUSTER. The ant must crawl so that if the plates were flattened out into a single rectangle as shown his path would be a line segment.



Answers to Exercises 8-2b

1. 85 cu. ft.
2. 20 cu. ft.
3. The number of cubic units of volume in a rectangular prism is the product of the number of square units of area in its base and the number of linear units in its height.
4. 2 ft.
5. 20 sq. ft., 2880 sq. in., $2\frac{2}{9}$ sq. yds.
   If x stands for the number of square feet in the end (base) the number sentence is $13x = 260$.
6. Volume of room is 1600 cubic feet.
   30 children would require 1500 cubic feet, so 30 children is a legal number.
   Greatest legal number of children is 32.
7. $V = Bh$

Answers to Exercises 8-2c

1. Volume is 748 cu. in.
   Area of wax paper is 431 sq. in.
2. No. Volume of 1st trunk is only 9 cu. ft.
   Yes. Volume of 2nd trunk is 10 cu. ft.
3. The number of cubic units of volume of a rectangular prism is the product of the numbers of linear units in the length, width, and height.
4. $V = \ell wh$
5. The number of inches in each edge of a cubic foot is 12, so the volume is $V = 12 \cdot 12 \cdot 12 = 1728$, or 1728 cu. in. There are 27 cubic feet in a cubic yard.

6. (a) Volume is 27 cubic inches.
   (b) Larger
   (c) No, if ℓ is less than 1, it is false.

7. $\frac{1}{2}$ inch

8. Yes, the volume is $23 \cdot 37 \cdot 59$ cubic inches.
   Volume of new prism is $46 \cdot 74 \cdot 118$ cubic inches.
   $46 \cdot 74 \cdot 118 = 2 \cdot 23 \cdot 2 \cdot 37 \cdot 2 \cdot 59 =$
   $2 \cdot 2 \cdot 2 \cdot 23 \cdot 37 \cdot 59 = 2^3 \cdot 23 \cdot 37 \cdot 59$
   Second volume is 2^3 or 8 times the first.

9. If all measurements are tripled the new volume will be 27 times the old volume $(27 = 3^3)$.
   If two dimensions are doubled and one tripled the new volume will be 12 times the old volume $(12 = 2^2 \cdot 3)$.

10. Weight $\approx$ 40,500 lbs. or $20\frac{1}{4}$ tons.

11. Volume $\approx$ 189,000 cu. ft., 7000 cu. yds.

12. 2 min.

13. $\frac{1}{16}$ mi. is the same as 110 yds. It is less than the distance between the end lines of a college football field (120 yds) but more than the distance between goal lines (100 yds). Number of people: 133,100.

14. Depth $\approx 1\frac{1}{2}$ ft., 18 in.

15. Weight of gold is 2250 lbs. Men can lift 2000 lbs., so they could not lift the chest.

16. 1728 in., 144 ft., 48 yds.

17. There would be space for 545,177,600 people. You could accommodate the population of the United States or of China, but not both at once.
    1 cu. mi. = 5,451,776,000 cu. yds.

18. $183\frac{3}{4}$ cu. in.

19. $10\frac{1}{4}$ and $10\frac{3}{4}$; $4\frac{3}{4}$ and $5\frac{1}{4}$; $3\frac{1}{4}$ and $3\frac{3}{4}$.

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20. $10\frac{1}{4}$ in. by $4\frac{3}{4}$ in. by $3\frac{1}{4}$ in. (smallest)
$10\frac{3}{4}$ in. by $5\frac{1}{4}$ in. by $3\frac{3}{4}$ in. (largest)
Volume of smallest prism is $158\frac{15}{64}$ cu. in.
Volume of largest prism is $211\frac{41}{64}$ cu. in.
Difference between smallest possible true volume, and answer to Problem 19 is $25\frac{33}{64}$ cu. in.
Difference between largest possible true volume and answer to Problem 19 is $27\frac{57}{64}$ cu. in.
Greatest possible error in answer to Problem 19 is, therefore, $27\frac{57}{64}$ cu. in. The volume is $(183\frac{3}{4} \pm 27\frac{57}{64})$ cu. in.

Answers to Exercises 8-2d

2. 154 sq. ft. of carpet. Volume 1386 cu. ft.
3. Volume of pantry is 180 cu. ft., volume of deep freeze is 42 cu. ft. Remaining space is 138 cu. ft. or $5\frac{1}{9}$ cu. yd.

Answers to Exercises 8-2e

1. (10)(10)(10) = 1000. There are 1000 cubic millimeters in 1 cubic centimeter.
2. (100)(100)(100) = 1,000,000. There are 1,000,000 cubic centimeters in 1 cubic meter.
3. (1000)(1,000,000) = 1,000,000,000. There are 1,000,000,000 cubic millimeters in 1 cubic meter.
4. (3)(3)(3) = 27. The volume of the 3 centimeter cube is 27 cubic centimeters. The volume of the 3 centimeter cube is larger than 3 cubic centimeters.

5. $2\left[(2)(3) + (2)(1) + (3)(1)\right] = 2\left[6 + 2 + 3\right] = 2(11) = 22.$
   The total surface area is 22 square meters.
   The total volume is 6 cubic meters.

6. (a) $(5)(4)(3) = 60$. The volume of the room without the closet is 60 cubic meters.
   (b) $(1)(1)(3) = 3$. The volume of the closet is 3 cubic meters.
   (c) (a)-(b) $60 - 3 = 57$. The difference in volumes is 57 cubic meters.
   (d) Yes.

8-3. Other Measures

Students should not be left with the impression that length is the only item that is measured. This chapter has dealt largely with length, area and volume. Some time was spent on angle measure and students are familiar with time, temperature, and weight as measured quantities. The limitation of time forces a choice between a casual and inadequate treatment of many measures and a more thorough treatment of a few. A real understanding of measurement concepts should make any future work with other measures simpler. To overcome the impression that only lengths, areas and volumes are measured, a short section on time and weight is included.

The distinction made in the student text between computing with numbers and computing with measurements should be emphasized though not belabored. Note that we actually give new meanings to "equals," "add," etc. This type of procedure is quite common further along in mathematics.

Seventh grade students have had some experience adding and subtracting measured quantities. The explanation given here should be treated as review. Subtraction provides an excellent opportunity to strengthen the student's understanding of place value and the base of a number system. This can be brought out

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by showing that in subtracting measures, a procedure similar to subtracting numbers is used. In base ten, when a digit in the minuend is too small for the step in subtraction a one in the next larger place is exchanged for ten of the smaller. Similarly, when necessary in subtracting measures, one large unit is exchanged for its equivalent in smaller units. The complete lack of system in the relationships between various units of measure offers an opportunity to review number bases by showing what bases would be convenient to use for a specific set of two related units. This should appeal to the brighter student.

Multiplication and division of measured quantities by a number will be new for most 7th grade students, but should cause little difficulty. Some of the better students will benefit from dividing composite quantities without changing to the smallest unit. An interpretation of the meaning of the remainder should be required. Practice in simple conversions is an essential part of this process, since remainders must be changed to the next smaller unit in order to complete the problem.

Some of these conversions provide considerable practice in fundamental arithmetic operations; units of area and volume require large numbers.

You will notice that without additional comment we have referred to the gram as the unit of mass, not weight, in the metric system. It was thought best not to involve the pupils in a full-scale discussion of this point here. An adequate treatment of the ideas belongs in a high school physics course. It is likely, however, that some of your pupils will ask questions about this terminology, and you will want to know how to answer them correctly, if not in complete detail. The following discussion should be adequate for this purpose. Should you wish more information on the subject, you might refer to Physics, Volume I, prepared by the Physical Science Study Committee of Educational Services, Inc. (This is the first volume of the so-called M.I.T. course for high school physics students.)

What is the weight of an object? It is a measurement of the force or "pull" of gravity on that object. An ordinary bathroom scale measures this pull by the amount it stretches, or twists, a spring. We think of weight as measuring the "quantity of matter" in an object, in some sense. A box of lead weighs more than the same box filled with feathers because the lead has a greater "quantity of matter" packed into the given volume than do feathers. There is another way to measure the "quantity of matter" of an object. This is to compare the object with some standard, or unit, bodies on a balance. If we have a supply of identical objects called "grams" we can determine the number of these "grams" it takes to balance the box of lead. This number of grams we call the mass of this much lead.

These two different ways of measuring "quantity of matter" can be used interchangeably for most purposes in any one fixed location, but they are not, strictly speaking, measurements of the same thing. Weight depends on the nearness to the center of the earth. The pull of the earth--gravity--on the box of lead would be much smaller in a space ship as far from the earth as is, say, the moon. The weight of the lead would be much smaller there. However, the lead would balance the same number of "grams" on the space ship which it balanced on the earth (the "grams" would themselves weigh correspondingly less) so its mass would be unchanged. To summarize:

> Weight is the pull of the earth. It changes as the distance between the object and the center of the earth changes.
>
> Mass is a comparison of the object with a set of unit bodies. It does not depend on the position in space where it is measured.

We humans are normally restricted to a very narrow range of altitude above sea level, and with that restriction, we can think of weight and mass as having a definite fixed relationship. (It is tempting to predict that as we enter the space age and are released from these restrictions, weight and mass and the distinctions between them will become subjects for household discussion.) In the English System weight is measured in pounds, mass in slugs. An object which has a mass of 1 slug has a weight of approximately 32.2 pounds at sea level. The weight in pounds of any object is approximately 32.2 times its mass in slugs. The more common unit in this system, of course, is the pound. In the metric system, when mass is measured in grams, weight is measured in dynes. An object whose mass is one gram has a weight of approximately 980 dynes at sea level. The weight of this same object in the English System would be approximately 0.0022 pounds. As you probably know, the more familiar unit in the metric system is the unit of mass, the gram.

In the text we have introduced only the more familiar units, pound and gram. This has made it necessary to use both words, mass and weight. You must judge for yourself how much of the above discussion of the two ideas you will use in your classroom. If the subject does come up, however, be sure to make one point: both mass and weight can be measured in either of the systems of units, English and metric. If you fail to point this out to the pupil, he may interpret the discussion in the text to mean that weight is something measured in the English system and mass something measured in the metric system.

Answers to Exercises 8-3a

1. (a) $\frac{231}{4}$ cu. in. or $57\frac{3}{4}$ cu. in.
   (b) Yes
   (c) Often there is a roof-shaped top containing the pouring spout but not filled with milk.

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2. Less than a quart by $2\frac{59}{64}$ cu. in. (Volume of container is $54\frac{53}{64}$ cu. in.)
3. The same volume of different materials have different weights. The old saying is roughly true for water.
4. (a) $64\frac{111}{128}$ cu. in. (Remember, however, this result is not as precise as the form makes it look).
   (b) $67\frac{3}{8}$ cu. in.
   (c) No
5. (a) $35\frac{5}{32}$ cu. in.
   (b) $33\frac{11}{16}$ cu. in.
   (c) The box holds $1\frac{15}{32}$ cu. in. more than it should.
6. Presumably in measuring dry quantities such as berries and the like there are air spaces not filled with anything, so this is compensated for by increasing the total volume which is to be called a quart.
7. Save $1.07 by buying the bushel.
8. Save $0.25 by buying the bushel.
9. Height should be $9\frac{3}{7}$ in. $\frac{7}{2} \cdot \frac{7}{2} \cdot h = \frac{231}{2}$
10. 11 in. by 7 in. by 3 in.

Answers to Exercises 8-3b

1. (a) 6 hr., 45 min.
   (b) 405 min.
   (c) $50\frac{5}{8}$ min. However, it should be pointed out that "passing time" would make these periods shorter.
   (d) $57\frac{6}{7}$ min.

2. 1269 hr.
3. 10 days
4. 4 hrs.
5. $218\frac{3}{4}$ lb.
6. Brand B. 2 oz. more.
7. (a) 420 oz.
   (b) $26\frac{1}{4}$ lb.
   (c) 15 cans
   (d) 14 cans
   (e) Brand B. Brand A costs $6.30. Brand B costs $6.16. Thus Brand B costs 14¢ less.
8. (a) 8 Tons
   (b) 16,000 lb.
9. (a) 1 million grams
   (b) 1,000 kg.
10. BRAINBUSTER. A cubic foot of water. Water weighs approximately $62\frac{1}{2}$ lbs. per cu. ft. while ice weighs about $57\frac{1}{2}$ lb. per cubic foot. Seventh grade students are not expected to be aware of specific gravity but some may be aware of the fact that ice floats.

Answers to Exercises 8-3c

1. 10 hrs. 10 min.
2. 16 yds. 11 in.
3. 15 gal. 3 qts.
4. 28 hrs. 6 min. 8 sec.
5. 13 sq. yd. 3 sq. ft. 119 sq. in.

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6. 11 cu. yd. 3 cu. ft. 807 cu. in.

7. (a) 22; 2 ft. 5 in.

   (b) One ten was exchanged for 10 ones; one foot was exchanged for 12 inches. In base ten, "1" in any place can be exchanged for ten in the next smaller place. In measures, one large unit can be exchanged for its equal in smaller units.

8. 2 yd. 2 ft. 4 in.

9. 1 hr. 35 min.

10. 2 gal. 1 qt. 1 pt.

11. 2 yd. 2 ft. 7 in.

12. 2 sq. ft. 79 sq. in.

13. 6 cu. yd. 1717 cu. in.

14. 31 hr. 30 min.

15. 121 yd. 2 ft. 9 in.

16. 1146 gal. 3 qt.

17. 79 hr. 38 min. 45 sec.

18. 102 T. 804 lb.

19. 728 sq. ft. 46 sq. in. or 80 sq. yd. 8 sq. ft. 46 sq. in.

20. 47 min.

21. 2 yd. 1 ft.

    Division without changing units.

```
          2 yds.         1 ft.
   10   )23 yds.         1 ft.
         20
         --
          3 yds.   =     9 ft.
                        ---
                        10 ft.
                        10 ft.
                        --
```

22. 203 lb.

23. 2 qt. 1 pt.

24. 39 sq. in.

25. 58 cu. in.

26. (a) Pints and quarts
    (b) Feet and yards
    (c) Ounces and pounds
    (d) Hours and minutes

27. BRAINBUSTER. 3 oz. ($\frac{3}{4}$ oz. is $\frac{1}{4}$ of the total weight; $4 \times \frac{3}{4} = 3$.)

Sample Questions for Chapter 8

Select from this list. There are too many questions for one test.

I. Multiple Choice, Completion and Matching

(b) 1. A hall is 6 feet long and $2\frac{1}{2}$ feet wide. How many square yards of carpet will cover it?

(a) 17 (d) $5\frac{2}{3}$
(b) $1\frac{2}{3}$ (e) None of these
(c) 15

(c) 2. The volume of a 4-inch cube is:

(a) The same as 4 cu. in.
(b) Smaller than 4 cu. in.
(c) 16 times as large as 4 cu. in.
(d) 4 times as large as 4 cu. in.
(e) None of these

(b) 3. 2 cu. ft. are equal to:

(a) 24 cu. in.

(b) 3456 cu. in.

(c) 266 cu. in.

(d) 18 cu. yd.

(e) None of these

4. Choose from the right-hand column, the term which best describes each term in the left-hand column and write its number on the line.

7 (a) Face of a cube ____

2 (b) Side of a rectangle____

1 (c) Side of an angle____

3 (d) Intersection of edges of a rectangular prism____

6 (e) Face of a rectangular prism____

1. Ray
2. Line segment
3. Point
4. Line
5. Plane
6. Rectangle
7. Square

6, 12,8 5. A rectangular prism has_____faces,_____edges, and _____vertices.

3 2 6. A rectangular solid is_____dimensional while any one of its faces is_____dimensional.

4 7. If the length and width of a rectangle are doubled, the area of the new rectangle is_____times that of the original one.

8. Which of the units in the right-hand column would be the best to use to measure the thing listed in the left-hand column? (A unit may be used more than once.) Write the number of the unit in the right-hand column on the line in the left-hand column.

4 (a) Air space in a room._____

8 (b) Linoleum needed to cover a shelf._____

1. Degree
2. Foot
3. Square foot

4 or 9 (c) Amount of water in a small aquarium.____

2 or 7 (d) A clothes line.____

4 (e) Space in a refrigerator.____

1 (f) An angle.___

4. Cubic foot
5. Mile
6. Square mile
7. Yard
8. Square inch
9. Cubic inch

Any two lengths that add to 13'. Areas will vary. 9. Jim has 26 feet of left-over fencing to use around a small garden. Give two different sets of dimensions he could use. Find the area of the garden in each case.

10. A rectangular prism is 8 ft. long, 2 ft. wide and 3 ft. high. The area of the largest face is____. sq. ft. The volume of the prism is_____cu. ft.

24 sq. ft.
48 cu. ft.

2592 sq. in. 2 sq. yd. 12. An area is found to be 18 sq. ft. This is the same as____sq. in. or____sq. yd.

II. Problems.

1. (a) Draw a rectangle 2$\frac{3}{8}$ in. by 3 in.

10$\frac{3}{4}$" (b) Find the perimeter of this rectangle.

7$\frac{1}{8}$ sq. in. (c) Find its area.

10 cu. ft. 2. A sand box is 3 feet wide and 4 feet long. How many cubic feet of sand are needed to fill the box to a depth of 10 in.?

9 in. 3. An aquarium is 14 inches wide, 22 inches long and holds 12 gallons of water. How deep is the water? (1 gal. = 231 cu. in.)

*4. A rectangular playground is 180 ft. by 330 ft.

59,400 sq. ft. 6,600 sq. yd. (a) What is its area in square feet? In square yds?

1020 ft. (b) What is its perimeter?

$5,940 (c) What would it cost to put blacktop on the playground at 90¢ a square yard?

$1552.50 (d) A fence is to be put around the two short sides and one long side of the playground. What would this fence cost at $2.25 per foot?

5. A chest is 30 inches wide, 2 feet high, and 5 feet long.

55 sq. ft (a) Find the area of the surface.

No (b) A small can of stain will cover 30 square feet. Is one small can enough to stain the top and sides?